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MATHEMATICS
FINANCE

The
MATHEMATICS
of
FINANCE

BY

[illegible] PERKINS

Chairman, Department of Mathematics
Middlebury College

AND

MARION PERKINS

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AND
RUTH MARION PERKINS

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PREFACE

[illegible] both [illegible]
[illegible] of the [illegible]
[illegible]
[illegible] helpful [illegible]
[illegible]

Llewellyn R. Perkins
Ruth M. Perkins

Middlebury, Vermont
June, 1941

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[illegible] 10,000 [illegible] D_x, S_x, a_x [illegible]

[illegible] A_x, R_x [illegible]

[illegible] 1000 P_x [illegible]

[illegible]

[illegible] Factor [illegible]

[illegible] Exact [illegible] between Two Dates [illegible]

[illegible] which Day [illegible] Date Falls [illegible]

[illegible]

[illegible] tant parts of the text.

[illegible] mathematics of finance.

INTRODUCTION

[illegible] that he will [illegible]

[illegible] to work in business life.

[illegible]

as hopeless.

[illegible]

examples finished below

Quotient

Dividend

28.96
6.096
5.792
304
290
14
14

1840
1536
304
269
35
34

[illegible]

retained.

[illegible]

[illegible]

CHAPTER I

[illegible]

[illegible]

[illegible]

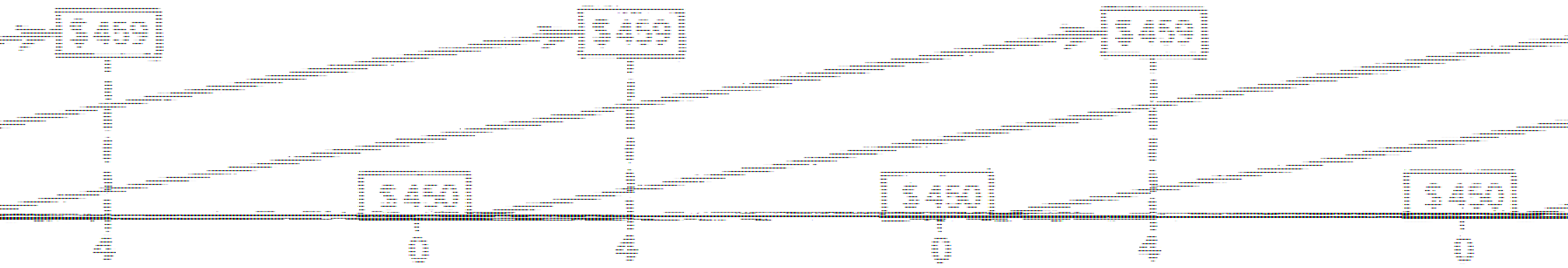
work.

[illegible] is $350.

[illegible] date to be computed?

[illegible] Simple Interest

[illegible]


$459
$450
0
4

Study of the solution of the numerical problem just given clearly shows that the amount of $1 for the given time was first found by adding to $1 the product of the rate of interest and the time expressed as the fraction of a year. This amount of $1 was then multiplied by the number of dollars loaned, giving the total amount. Hence we may employ the literal symbols and write:

$$S = P(1 + ni) \tag{I·1}$$

Moreover, we saw that the simple interest for the term is the amount less the principal, or, in symbols:

$$I = S - P = P(1 + ni) - P = Pni \tag{I·2}$$

These two formulas enable us to solve any problem in simple interest. It should be noted that there are five different quantities involved in these two formulas. Hence it is possible to find the value of any two of these quantities as soon as the values of three of them are known.

The letter n is defined as representing the time for which interest is to be computed expressed as a fraction of a year. The denominator of this fraction is a year expressed as days and the numerator is the number of days in the interest term. Taking 365 days to the year gives a slightly less simple fraction with which to compute than taking 360, or 12 months of 30 days each. For that reason the custom grew of using the less accurate but more convenient fraction with 360 as its denominator. This gives what is commonly called *ordinary interest* but many institutions use the more accurate form and obtain *exact interest*. It would be logical to use with the denominator 360 the length of the term of interest found by counting the number of entire months in it, multiplying by 30 to reduce this to days and finally adding the remaining days in the term. With the denominator 365 the exact number of days in the term is used.*

---

* In this matter of computing time some strange anomalies are to be found. Perhaps the most important of these is the result of the following ruling of the governing board of the New York Stock Exchange: "COMPUTATION OF INTEREST: Interest at the rate specified in the bond dealt in 'and interest' shall be computed on the basis of a 360-day-year, i.e., each calendar month shall be considered to be $\frac{1}{12}$ of 360 days or 30 days and each period from a date in one month to the same date in the following month shall be considered to

[illegible]

bankers' rule.*

[illegible]

EXERCISES

[illegible]

XI and XXI.

[illegible]

[illegible]

To From

November 1, 1938? What principle does this wording of the problem bring into the foreground?

4. Restate Ex. 1 so as to make it an evident illustration of the principle of simultaneity in all three of its parts.

5. A man who had borrowed \$1200 at 4 per cent found when he came to repay the loan that he owed \$1208.10. How long a time did it take the \$1200 to become \$1208.10? *Ans.* 2 months, 1 day

6. Solve formula I·1 for P, n and i respectively. Do the same for formula I·2.

7. Eliminate i between the two formulas I·1 and I·2 and solve the resulting formula for each letter in it. Put the results into words.

8. If e represents the exact number of days between two given dates and a represents the approximate number of days computed by counting the months as having 30 days each, then express the exact, the ordinary, and the bankers' rule interest on P dollars at the annual rate, i.

9. Find the ordinary and the exact simple interest on \$4560 from January 7, 1934, to September 27, 1934, at 6 per cent, making use of the table giving interest at 1 per cent on \$1000 (see Table XI).

10. Work Ex. 9 independently of the table and compare the difficulty of the two solutions.

11. In computing the exact interest called for in Ex. 1, is August 4, 1938, counted as one day of the interest period? Is November 12, 1938? Will these same answers apply when computing the time for the ordinary interest? For banker's rule interest?

12. How long will it be necessary to keep \$600 loaned out at ordinary simple interest at 3 per cent to have it accumulate to \$606?

13. If money can be loaned in Middlebury at simple interest at 4 per cent ordinary interest, to what sum would \$500 on May 9, 1938, be equivalent on October 16, 1938? What principle does the wording of this problem bring into the foreground?

§ 2. **Compound Interest.** Money at work is like man at work in that it produces value by its labors. Oftentimes its labor is seasonal with a period of stagnation between seasons. When money is borrowed for seasonal labor it is usually borrowed for a few months, and when the fruits of its labor are harvested the borrower pays back the loan together with a part of this harvest as simple interest on the loan. This is perfectly fair for this type of loan.

But when Mr. Smith borrows money from Mr. Brown and uses it to labor for him over something from which he gathers his harvest of value once per year it is only fair that he should pay Mr. Brown interest once per year, or if he continues to use Brown's money for a longer time he should, at the end of each year, add the interest earned during that year to the original borrowing and

[illegible]

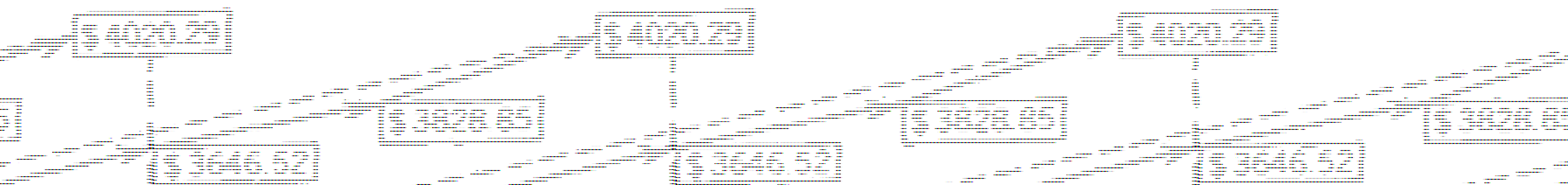
occur.

[illegible] the following problem [illegible]

[illegible]

[illegible] next is found to be:

[illegible]


$ 4020.29
$ 3828.65
$ 3646.52

[illegible] ut into general form.

[illegible]

ually

[illegible]

oftener

[illegible]

years

[illegible]

term

[illegible]

term

[illegible] formulas are obtained:

$$S = P(1 + i)^n \quad [1 \cdot 3]$$

$$P = S(1 + i)^{-n} \quad [1 \cdot 4]$$

[illegible] $\left(1 + \frac{j}{m}\right)^m$ [illegible]

[illegible]

$$S = P\left(1 + \frac{j}{m}\right)^{mn} \quad [1 \cdot 5]$$

EXERCISES

Ans. $4000

compound-interest rate.

$(1+i)^n$. This gave the sum of money at the beginning which would accumulate to a given sum, S, if put at annually compounded interest at rate i for n years. In other words, given a sum of money at interest at a given compound-interest rate the sum equivalent to it at any *previous* time can be found by multiplying this sum, S, by $\frac{1}{(1+i)^n}$. Verify this by solving formula I·3 for P. If the fraction in this result, $\frac{1}{(1+i)^n}$, is replaced by v^n and P is given the broader definition of being the present or the previous value of S, we have a very useful formula:

$$P = Sv^n \text{ (See Table II for } v^n\text{)} \qquad [\text{I}\cdot 8]$$

by which the equivalent value of a sum of money at compound interest can be computed for any previous time, n years before the time of the value S. The quantity, v, is called the discount factor and is evidently defined by the equation:

$$v = \frac{1}{(1+i)} = \frac{1}{\left(1+\frac{j}{m}\right)^m} \qquad [\text{I}\cdot 9]$$

§ 3. Combining Several Transactions. Frequently when two individuals are conducting business with each other over a period of time it becomes desirable to combine a series of transctions of different dates and strike a balance so that by a single transaction they may simplify their financial relations. Consider the following simple case.

PROBLEM. Brown made Smith the following loans: January 20, 1936, \$500 at 5 per cent annually compounded interest for 5 years; January 20, 1938, \$1600 for 6 years at 4 per cent semiannually compounded interest. Either Smith or Brown can now hire money in the open market at $3\frac{1}{2}$ per cent effective. (a) If \$2384.89 paid by Smith now will in fairness to both Brown and Smith cancel both debts what is the present date? (b) What sum would equitably balance both these debts on January 20, 1939?

SOLUTION. (a) The amount of the first debt at maturity, January 20, 1941, is

$$S_1 = 500(1.05)^5 = 500 \times 1.276281 = \$638.14$$

The amount of the second debt at its maturity, Jan. 20, 1944, is

$$S_2 = 1600(1.02)^{12} = 1600 \times 1.2682418 = \$2029.19$$

[illegible]

after January 20, 1936 [illegible]

$$\frac{[illegible]}{(1.035)^{8}}$$

[illegible]

$$\frac{2029.19}{(1.035)^{8}}$$

[illegible]

hence,

[illegible]

Dividing through by 2384.89

$$\frac{2078.29}{2384.89} = .871442$$

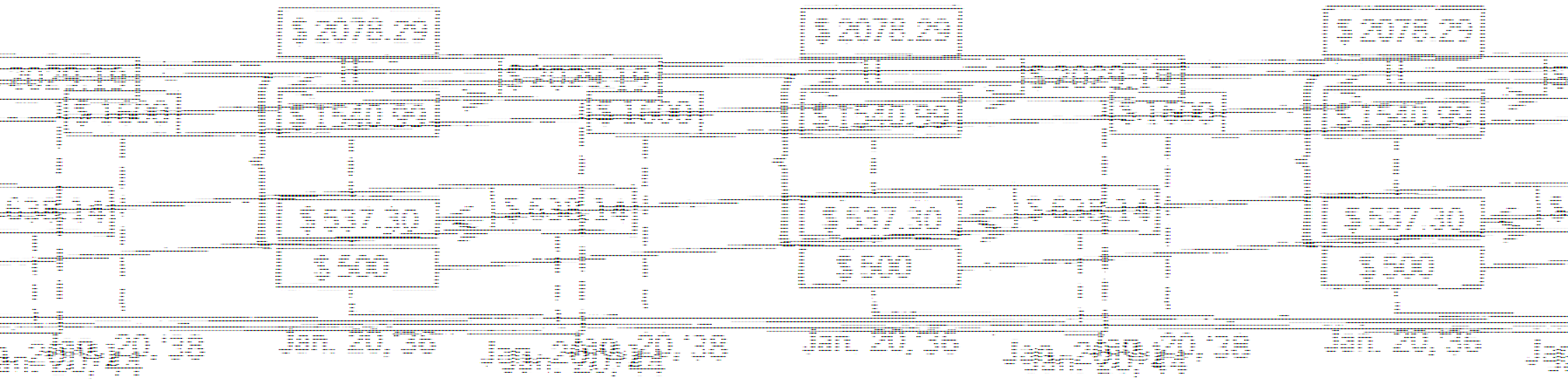
[illegible]

use of logarithms

$$x \log v = \log (.871442)$$

hence we get

[illegible]


$ 2078.29
$ 1540.99
$ 1000
$ 537.30
$ 500
Jan. 20, '36
Jan. 20, '38

ary 20, 1939.

alized as follows:

[illegible] interest

[illegible]

[illegible] or Y is the [illegible]

[illegible]

[illegible] of debts.

[illegible] equation of value, [illegible]

EXERCISES

[illegible] resulting.

[illegible]

[illegible]

[illegible] their combine[illegible]

[illegible]

[illegible] compounded [illegible] internally? compounded [illegible]

[illegible] money for [illegible]

[illegible]

[illegible] shorter period [illegible]

[illegible]

and of val[illegible] equation of time [illegible]

[illegible] Continuously Compounded [illegible]

[illegible] Equation [illegible]

[illegible] times [illegible]

[illegible]

[illegible]

[illegible] find [illegible] compounding [illegible]

[illegible] continuous [illegible] infinite [illegible]

[illegible]

[illegible] binomial [illegible]

orem into

[illegible]

[illegible] taken, drop out leaving [illegible]

$$= 2.7181818 = e$$

[illegible]

[illegible] equation I·5 becomes

$$S = Pe^{jm} \quad [I\cdot 13]$$

[illegible]

time.

[illegible]

$$i = e^{j} - 1 \quad [I\cdot 14]$$

[illegible]

6 per cent, 5 per cent.

[illegible]

$(1 + .02)^4$.

[illegible]

[illegible]

discount rate.

[illegible] is often very helpful:

[illegible] other promise

[illegible] as follows:

$$P = F(1 - nd)$$

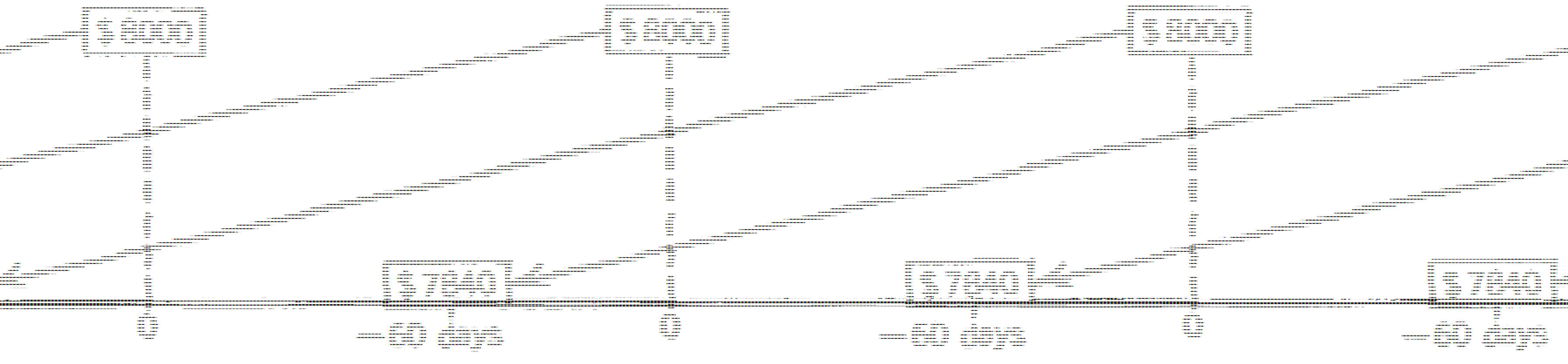
[illegible]

[illegible] discount.

[illegible] the discount.

[illegible]

8000 − 7840 = [illegible]


$8000
$7840
0
−60 days

[illegible] by simple discount.

EXERCISES

[illegible] merchant can show the [illegible]

[illegible]

[illegible] operation, the discount [illegible]

[illegible] discount

[illegible] and

[illegible] pounded

then

$$P = F(1 - d)^n \qquad [I \cdot 18]$$

and

$$D = F - P \qquad [I \cdot 19]$$

When the discount is [illegible] more than once a year, [illegible] $\left(1 - \frac{f}{m}\right)^m$ [illegible] different form for I·18,

$$P = F\left(1 - \frac{f}{m}\right)^{mn} \qquad [I \cdot 20]$$

receive as discount?

is used.

proceeds

$ 2000

$ 1767.20

of compound

ceeds be?

of F in formula I·20

By formula I·19

$ 3391.75

[illegible] greater accuracy is needed. [illegible]

EXERCISES

[illegible] both (1) and (2).

[illegible] in this case?

[illegible] curing the money?

[illegible]

[illegible]

when $f = j$,

$$\lim_{m \to \infty} \left(1 + \frac{j}{m}\right)^m \Big/ \left(1 - \frac{f}{m}\right)^{-m} = [illegible]$$

[illegible]

Taking the nth root and [illegible]

$$i = \frac{1}{(1-d)} \quad [\mathrm{I}\cdot 21]$$

In the [illegible] money invested.

[illegible] discount [illegible] compounded [illegible] $\left(1-\frac{f}{m}\right)^{m}$ [illegible] the formula becomes:

$$i = \frac{1}{\left(1-\frac{f}{m}\right)} \quad [\mathrm{I}\cdot 22]$$

[illegible] P_{nd} [illegible] giving [illegible]

$$i = \frac{d}{1-nd} \quad [\mathrm{I}\cdot 23]$$

In the case of a [illegible] discount [illegible] evident [illegible] effective interest [illegible] depend upon the [illegible] Since [illegible] approaches [illegible] equivalent interest [illegible] given discount rate [illegible]

2. A banker invests $10,000 in discounting a 6-month note at a rate of 6 per cent and $10,000 in an interest loan on which the interest rate is also 6 per cent. How many dollars does he make in 6 months on each transaction? Make a definite statement in comparison of the two transactions.

3. Compare the interest rate earned by the discount transaction with that in the interest transaction in Ex. 2.

4. Three notes, each for $1000, one due in 60 days, one due in 90 days, and one due in 6 months, are discounted for their full period at the same yearly discount rate, 6 per cent. Find the proceeds of each, the discount on each, and the equivalent yearly simple-interest rate that would give the same return on the investments.

5. John Smith gave Henry James his note for $3000 to be paid in 4 years with semiannually compounded interest at 6 per cent. James has an idea that he can discount the note at once at a discount rate so low that he can reloan the proceeds at the same rate charged Smith and so make a still higher rate on his money. What would be the highest rate of discount that he could pay and still make money on such a double loaning transaction?

6. A man who had an unfortunate experience in collecting on a long-term loan decided never to loan his money where he could not collect at the end of 6 months. He had been getting 5 per cent effective on his long-term loans. After another term of 5 years he found that his money could be kept at interest in such short-term loans only 85 per cent of the time. What rate of interest must he get to make these loans as profitable as the others?

7. A man made two loans of $100 each. On the first he received 6 per cent for the first 12 years and then lost the entire loan. On the second he received no return for 9 years and then 6 per cent for 4 years and finally full repayment of the loan. Which was the better of the two investments?

MISCELLANEOUS EXERCISES

1. Find the quarterly compounded interest rate that is equivalent to 6 per cent compounded semiannually. *Ans.* .05956

2. In 1614 Peter Minuet paid the Indians $24 for Manhattan Island. If they had deposited this sum at interest at an effective rate of 6 per cent to what sum would it have grown by the end of this year?

3. To what semiannually compounded rate of interest does a discount rate of 6 per cent for one year correspond? Is the discount simple or compound? *Ans.* .0628

4. The open-market rate of interest is 5 per cent effective and two debts are owed. One is for $2000 due in 4 years without interest and the other is for $1000 due in 10 years without interest. What lump sum now would discharge both debts equitably? *Ans.* $2269.32

5. On what date would $2500 pay both debts in Ex. 4 equitably?

6. If the $2000 debt had been at interest to the creditor at 4 per cent effective and the $1000 at 3 per cent semiannually what present lump sum would then have been needed to balance both debts at once?

7. In Ex. 4 what sum would be required to balance the account 2 years hence? *Ans.* $2490.90

MISCELLANEOUS EXERCISES

[illegible]
interest have been?

[illegible] What does it amount to?

[illegible]
Ans. 20 cents per year

[illegible]
draw the note? *Ans.* $2030.85

[illegible] interest rate (effective)

[illegible]

CHAPTER II

[illegible] PAID [illegible]

[illegible] *certain.*

[illegible] first payment is made.

[illegible] consideration.

[illegible] the following [illegible]

[illegible]

Now the accumulation of \$1 per year to the time of the last payment including the last payment is found as follows: the first payment of \$1, being paid at the end of the first year, is at interest for $(n - 1)$ years and by formula I·3 amounts to $1(1 + i)^{n-1}$ at the time of the last payment; the second yearly payment by the same reasoning amounts to $(1 + i)^{n-2}$, and so on until the next to the last payment which has only a year to run and amounts to $(1 + i)$, while the last payment is a cash payment and equals 1. The sum of all these amounts gives the amount of the annuity as $s_{\overline{n}|} = (1 + i)^{n-1} + (1 + i)^{n-2} + \cdots + (1 + i) + 1$, or in reverse order,

$$s_{\overline{n}|} = 1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-2} + (1 + i)^{n-1}$$

Note now that this series of terms forms a geometric progression with its first term 1, its common ratio $(1 + i)$ and its number of terms n. The sum of a geometric progression is the first term times the common ratio raised to the nth power less the first term and divided by one less than the common ratio. In this case, therefore, the sum is:

$$s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{(1 + i) - 1} = \frac{(1 + i)^n - 1}{i} \qquad [\text{II}\cdot 1]$$

This formula for the sum of the amounts of n payments, each accumulated to the time of the last payment, may be considered as the fundamental formula of annuities. All the other needed relations of annuities can be obtained from it very simply.

A dated-value graph of $s_{\overline{n}|i}$ may be made as follows:

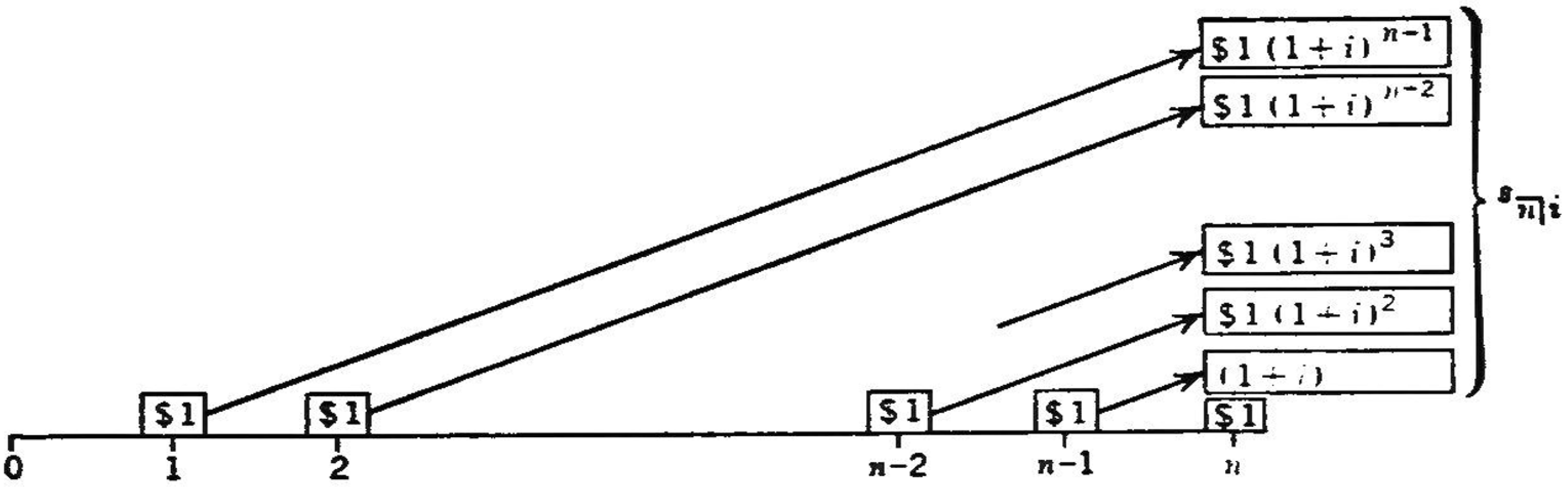


FIG. 7. A dated-value graph of the accumulation of a simple annuity certain of \$1 annual rent.

Since $s_{\overline{n}|i}$ is the amount or accumulation of an annuity which [illegible] dollars per year would [illegible]

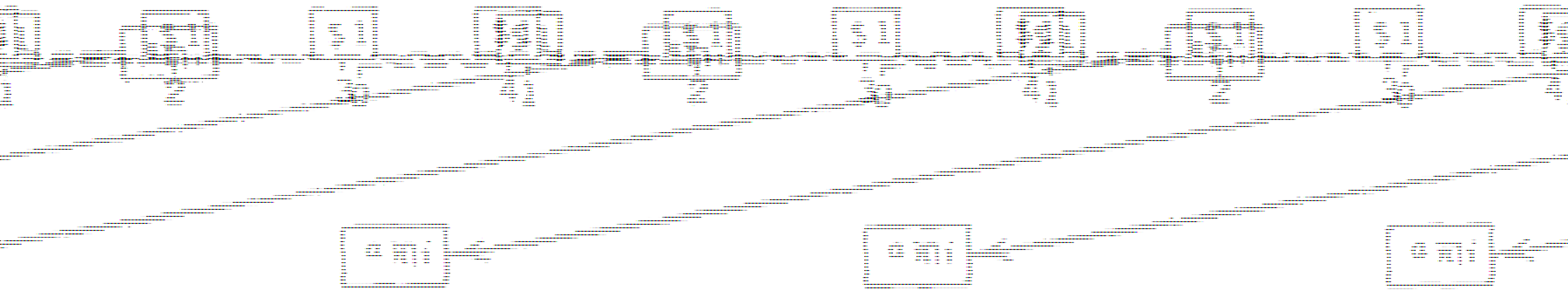
$$K = R \times s_{\overline{n}|i} \quad [\text{II}\cdot 2]$$

[illegible]

$$[illegible] = R\,\frac{(1+i)^n - 1}{i} \;[illegible]$$

[illegible] the beginning of [illegible]

$$a_{\overline{n}|i} = \frac{1 - v^n}{i} \quad [\text{II}\cdot 4]$$



[illegible] difference [illegible]

[illegible] the end of [illegible]

and many of these can be done with the necessary accuracy by the use of seven-place logarithm tables.

As was the case in the compound interest formulas of Chapter I, when the interest in an annuity is compounded oftener than once a year the i in the formulas so far derived is replaced by its equivalent, $\left(1+\frac{j}{m}\right)^m - 1$ (I·7 and Ex. 1, p. 14), and the $(1+i)$ is replaced by $\left(1+\frac{j}{m}\right)^m$ (I·7) and the resulting formulas reduced to a form that is best adapted to use with the tables.

For nominal interest rates, formula II·1 becomes:

$$s_{\overline{n}|j_m} = \frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\left(1+\frac{j}{m}\right)^{m} - 1} = \frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\frac{j}{m}} \times \frac{\frac{j}{m}}{\left(1+\frac{j}{m}\right)^{m} - 1}$$

$$= \frac{1}{s_{\overline{m}|\frac{j}{m}}} \times s_{\overline{mn}|\frac{j}{m}} \qquad [\text{II}\cdot 6]$$

and formula II·2 becomes

$$K = R \times \frac{1}{s_{\overline{m}|\frac{j}{m}}} \times s_{\overline{mn}|\frac{j}{m}} \qquad [\text{II}\cdot 7]$$

The same process of backward transference for n years or mn periods at the nominal rate would necessitate multiplying II·7 by $v^n = \frac{1}{\left(1+\frac{j}{m}\right)^{mn}}$ giving

$$A = R \times \frac{1}{s_{\overline{m}|\frac{j}{m}}} \times \frac{1}{\left(1+\frac{j}{m}\right)^{mn}} \times s_{\overline{mn}|\frac{j}{m}} = R \times \frac{1}{s_{\overline{m}|\frac{j}{m}}} \times a_{\overline{mn}|\frac{j}{m}} \qquad [\text{II}\cdot 8]$$

Finally, the value, L, of the whole annuity at any time t_1 years before the beginning or t_2 years after the end of the term of the annuity is

$$L = A \times \frac{1}{\left(1+\frac{j}{m}\right)^{mt_1}} = Av^{mt_1} \quad \text{or} \quad L = K\left(1+\frac{j}{m}\right)^{mt_2} \qquad [\text{II}\cdot 9]$$

as it fell due?

$$7.537626 = \$9045.15$$

$$.7651344 = \$6920.76$$

$$\frac{28943.52}{1500} = 19.29568$$

rate is $3\frac{1}{2}$ per cent

$$\frac{29291.03}{1500}$$

EXERCISES

1. A farmer moves to town and rents his farm for \$500 per year for a period of 10 years, payable at the end of each year. He can earn $3\frac{1}{2}$ per cent semiannually compounded by putting his rent into the savings bank. To what will his rental amount at the end of the 10 years if he does this? *Ans.* \$5874.01

2. To what would this income from his farm amount in 20 years from the beginning of the term of the rental if kept in the bank as described in Ex. 1? *Ans.* \$8310.42

3. If the renter of the farm in Ex. 1 were to pay his entire rent at the beginning of the 10 years how much would he have to pay?

4. A man promises to contribute \$100 yearly for 10 years to the Boy Scout organization. If he could have put his contributions at interest at 5 per cent compounded quarterly to what sum would his gift amount in the 10 years? *Ans.* \$1263.35

5. At what effective interest rate will an annual annuity of \$200 per year accumulate to \$5108.93 in 20 years?

6. Find the amount of an 8-year annual annuity of annual rent of \$800 if it is accumulated at a nominal rate of 4 per cent compounded quarterly. *Ans.* \$7387.26

7. Find the purchase price of the above annuity. *Ans.* \$5372.79

8. In what time will an annual annuity of \$500 increase to \$7958.58 at 5 per cent? *Ans.* 12 years

9. Had the annuity in Ex. 8 been accumulated at 4 per cent how long would it have taken to accumulate to the same amount?

10. Show that $\frac{1}{a_{\overline{n}|i}} - \frac{1}{s_{\overline{n}|i}} \equiv i.$

11. Solve formula II·3 for R and decide which table will make the solution of a numerical problem easiest when all except R are known quantities.

12. Suppose n were the unknown in formula II·3, see if you could devise a method for solving a numerical problem.

Early in the chapter it was pointed out that the formula $s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$ (II·1), is the fundamental formula of annuities in that all the others could be derived from it. This is true and very helpful in getting the formulas. In stating problems and discussing methods of solution another point of view is more helpful and time-saving. Note that there are two classes of formulas: those containing the factor $s_{\overline{n}|}$ and those containing the factor $a_{\overline{n}|}$.

If we write these symbols thus without any indication of the interest rate and its method of compounding it makes a general analysis easier to write, and when we approach the stage of actual computation we replace the indefinite symbol $s_{\overline{n}|}$ or $a_{\overline{n}|}$ by the more

[illegible]

[illegible] This [illegible] rates.

$$s_{\overline{n}|} = [illegible] \left(1 + \frac{j}{m}\right)^m \text{ or}$$

$$a_{\overline{n}|} = [illegible] \left(1 + \frac{j}{m}\right)^m \text{ or}$$

[illegible]

$$\times s_{\overline{n}|} \times (1 + [illegible]) = R \times s_{\overline{n}|}$$

[illegible]

EXERCISES

1. A man pays $450 per year in advance as rent for a house. How expensive a house could he have bought had he bought in the beginning and amortized his debt at 5 per cent effective with the rental money in 10 years?

2. A boy's life was insured by his father when he was 10 years old, $1000 of insurance was secured for an annual premium of $16 per year paid in advance. At 50 the boy was alive and well and wondered to what sum of money his premiums would have accumulated had he put them into a bank paying 4 per cent semiannually compounded instead of into the insurance.

3. If you had been that boy what arguments pro and con could you give for the insurance instead of the investment?

4. By the aid of the time-value graph show that (a) $s_{\overline{n}|} = 1 + \mathrm{s}_{\overline{n-1}|}$; (b) $a_{\overline{n}|} = \mathrm{a}_{\overline{n+1}|} - 1$ (c) Solve both (a) and (b) into forms useful with tables.

5. If you were asked to compute the amount of an annuity due of $5000 per year for 5 years at 4 per cent effective, would you use formula II · 12, or would you make use of the result of Ex. 4 (a) and use $K = R(s_{\overline{n+1}|} - 1)$?

6. Compute the amount of the annuity due in Ex. 5 by both methods.

7. Compute the present value of this same annuity due by two different formulas.

8. Make a dated-value chart for formulas II · 10 and II · 11.

§ 10. Deferred Annuities. If the payments of an annuity are postponed for a term of years after its purchase it is called a deferred annuity. The symbol for the purchase price of an n payment annuity of $1 per year deferred for m years is ${}_{m|}a_{\overline{n}|}$ and it is the purchase price of an annuity for $m + n$ payments less the purchase price of one for m payments.

$$ {}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|} \qquad [\mathrm{II} \cdot 14] $$

This value of the purchase price of $1 multiplied by the annual rent gives the purchase price of a deferred annuity of any annual rent. The amount of a deferred annuity of n payments is precisely the same as that of an annuity that has not been deferred.

An annuity due may be deferred also, but in this case the period of deferment is the number of years from the date of purchase to the date of the first payment instead of one year less than this, as in the case of a simple annuity certain.

For the annuity due:

$$ {}_{m|}\mathrm{a}_{\overline{n}|} = \mathrm{a}_{\overline{m+n}|} - \mathrm{a}_{\overline{m}|} \qquad [\mathrm{II} \cdot 15] $$

§ 11. Perpetuities. Frequently a library, museum or other public benefaction is accompanied by a sum of money with which

[illegible]

$$a_{\infty} = \frac{1}{i} \qquad [II\cdot16]$$

[illegible] [II·17] and as shown [illegible]

[illegible] solved for A_{∞} giving

$$A_{\infty} = R_r \times \frac{1}{i} [illegible] \qquad [II\cdot18]$$

[illegible]

This gives the capitalized cost of an annually paid perpetuity as:

$$C = R + \frac{R}{i} = R\left(\frac{1+i}{i}\right) = \frac{R}{d} \qquad [\text{II}\cdot 20]$$

The capitalized cost of an r-interval perpetuity is given by:

$$C_r = R_r + R_r \times \frac{1}{i} \times \frac{1}{s_{\overline{r}|i}} = \frac{R_r}{i}\left(i + \frac{1}{s_{\overline{r}|i}}\right) = \frac{R_r}{i a_{\overline{r}|i}} \qquad [\text{II}\cdot 21]$$

EXERCISES

1. Compare what has been said about capitalized cost with what has been said about the cost of an annuity due and see if you can devise and justify an alternative name for capitalized cost that will show the relationship between it and the annuity due.

2. What is the purchase price of a perpetuity of \$2000 if interest is 4 per cent effective? *Ans.* \$50,000

3. Suppose that the payments of the perpetuity in Ex. 2 had been deferred 5 years; what would have been the purchase price? *Ans.* \$41,096.36

4. A bridge with an estimated life of 40 years cost \$40,000 to build. What sum must be set aside at the time of building to care for renewals in perpetuity if interest is at 4 per cent semiannually compounded? *Ans.* \$10,321.40

5. A street costs \$12 per square yard to pave and the paving has an estimated life of 25 years. Find the capitalized cost per square yard to keep this street paved if money earns 5 per cent. *Ans.* \$17.03

6. A wealthy man, wishing to give an endowed library as a memorial to his mother, called in an architect who estimated the costs as follows: to build, \$100,000; for upkeep of building, \$10,000 a year; for books and attendants, \$12,000 yearly. If interest is at $3\frac{1}{2}$ per cent effective what sum must the man provide to finance the project?

7. A man died leaving his \$500,000 estate invested in a perpetuity paying 5 per cent. His will provided that his widow should receive the entire income until she had received one-half of the entire estate; then the entire income was to go in perpetuity to a hospital in which he was interested. How long did his widow receive the income? *Ans.* 14 years and \$5267.94 in the fifteenth year.

8. Starting with the relation $\frac{1}{a_{\overline{n}|}} - \frac{1}{s_{\overline{n}|}} = i$ show that $1 + \frac{1}{i s_{\overline{n}|}} = \frac{1}{i a_{\overline{n}|}}$.

9. Make a dated-value chart of II·18 and II·19.

MISCELLANEOUS EXERCISES

1. An insurance company writes endowment policies which give the insured an option between cash payment and an annuity. What annual payment of a 10-year annuity would make it the equivalent to a cash payment of \$1000 at the maturing of its policy if interest is at $3\frac{1}{2}$ per cent? *Ans.* \$120.24

2. A business block was leased for 99 years at an annual rental of \$6000. What is the value at the time of signing the lease of the last 10 years of the rent if interest is at $3\frac{1}{2}$? *Ans.* \$2335.64

[illegible] of which the present value [illegible]

[illegible] per cent [illegible]

[illegible]

[illegible]

[illegible] per cent $20,000? *Ans.* 3¾ per cent.

[illegible]

[illegible] for $20,000? [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] at the end of the year [illegible]

[illegible] payment $29.45 [illegible]

[illegible]

[illegible] principal [illegible]

[illegible]

[illegible]

[illegible]

[illegible] *Ans.* Mr. A's; $83.40

[illegible]

[illegible] annuity [illegible]

[illegible] value of the policy [illegible]

[illegible] each payment [illegible]

[illegible]

[illegible] the end of each year [illegible]

[illegible] 25 years [illegible]

[illegible]

[illegible]

12. Show that $a_{\overline{n-1}|} + 1$ [illegible]

13. Show that $s_{\overline{n+1}|} - 1$ [illegible]

[illegible]

[illegible] the time [illegible]

[illegible]

[illegible]

[illegible] how much? [illegible]

CHAPTER III

ANNUITIES PAID IN INSTALLMENTS

§ 12. **General Considerations.** Rent paid monthly or weekly, sickness or accident benefits paid weekly, wages paid daily, weekly, or monthly, installments on articles bought on the installment plan and many other transactions of a like nature in which equal equispaced payments are involved are called installment-paid annuities.

The same symbols used in the annually paid annuities of the previous chapter are also used for installment-paid annuities, with slight additions to prevent ambiguity, as in the case of the amount of 1, and the present value of 1 for which the symbols are changed to $s_{\overline{n|}}^{(p)}$ and $a_{\overline{n|}}^{(p)}$ to show that these represent respectively the amount and the present value of a $1 annuity paid in p installments per year.

The two symbols just given are general in the use defined and to make them complete the n is followed by either i or $j_{(m)}$ to indicate whether the rate is effective or nominal and to give its frequency of compounding.

Now the value of an n-year annuity of annual payment of $1 when the dollar is paid in p equal installments differs from the value of a similar annuity paid once a year because the p parts of the dollar constitute, in themselves, a sort of miniature annuity whose amount at the end of a year is the *actual* annual rent of the annuity and is always slightly more than the $1 specified as its rent, just as in compound interest the actual increase in $1 in a year is greater than the nominal rate when the interest is compounded oftener than once a year. The amount of this miniature one-year annuity composed of p payments made at intervals of $\frac{1}{p}$th of a year is symbolized by $s_{\overline{1|}i}^{(p)}$.

The value of $s_{\overline{1|}i}^{(p)}$ can now be found as follows: the first payment, $\frac{1}{p}$ of $1, will amount to $\frac{1}{p}(1+i)^{1-\frac{1}{p}}$ at the end of the year,

[illegible]

[illegible]

series is

[illegible]

[illegible]

[illegible] dollars per year would [illegible]

$$R_1 = R \times s_{\overline{1}|}^{(p)}$$

[illegible]

value or cost price is:

comparison with II·5:

EXERCISES

[illegible]

Ans. 8 full years

[illegible]

Ans. $323.91

[illegible]

Ans. $9380.49

[illegible]

$$K = R\frac{\left[\left(1+\frac{j}{m}\right)^{m}-1\right]}{p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1\right]}\times\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\left(1+\frac{j}{m}\right)^{m}-1}$$

$$= R\frac{1}{p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1\right]}\times\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{1}$$

$$= R\frac{\frac{j}{m}}{p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1\right]}\times\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\frac{j}{m}}$$

Considering the second fraction first, it is easy to see that we need only note that $\frac{j}{m}$ is an interest rate and then compare this form with formula II·1 to see that it is $s_{\overline{mn}|\frac{j}{m}}$.

The first fraction of this product requires more thorough discussion. It takes three different forms, according as $m = p$, $m > p$ or $m > p$.

If $m = p$,

$$\frac{\frac{j}{m}}{p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1\right]} = \frac{\frac{j}{m}}{m\left[\left(1+\frac{j}{m}\right)-1\right]} = \frac{1}{m}$$

If $m < p$, and m is a factor of p so that $\frac{p}{m} = k$ or $p = mk$, we see by comparison with III·1 that:

$$\frac{\frac{j}{m}}{p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}}-1\right]} = \frac{\frac{j}{m}}{mk\left[\left(1+\frac{j}{m}\right)^{\frac{1}{k}}-1\right]} = \frac{1}{m}\times s^{(k)}_{\overline{1}|\frac{j}{m}}$$

ALLMENT ANNU

parison with II·1 that

CASE I. When $m = p$,

which follows directly,

by multiplying by

and finally, from III·6

or

CASE II. When $m < p$,

$$A = R \times \frac{1}{m} \times s^{(k)}_{\overline{1}|\frac{j}{m}} \times a_{\overline{mn}|\frac{j}{m}} \text{ (Tables IX and IV)} \qquad [\text{III}\cdot\mathbf{11}]$$

and finally, from III·10 and III·11, $L = A \times v_{\frac{j}{m}}^{mt_1}$ or $K\left(1 + \frac{j}{m}\right)^{mt_2}$ as in Case I, except that the values used for K and A are those obtained by using III·10 and III·12.

Case III. When $m > p$ and p is a factor of m so that $\frac{m}{p} = h$,

$$s^{(p)}_{\overline{n}|j_m} = \frac{1}{p} \times \frac{1}{s_{\overline{h}|\frac{j}{m}}} \times s_{\overline{mn}|\frac{j}{m}} \text{ (Tables V and III)} \qquad [\text{III}\cdot\mathbf{12}]$$

From which follows directly

$$K = R \times \frac{1}{p} \times \frac{1}{s_{\overline{h}|\frac{j}{m}}} \times s_{\overline{mn}|\frac{j}{m}} \text{ (Tables V and III)} \qquad [\text{III}\cdot\mathbf{13}]$$

and, by multiplying by $v_{\frac{j}{m}}^{mn}$,

$$A = R \times \frac{1}{p} \times \frac{1}{s_{\overline{h}|\frac{j}{m}}} \times a_{\overline{mn}|\frac{j}{m}} \text{ (Tables V and IV)} \qquad [\text{III}\cdot\mathbf{14}]$$

And finally, from III·14 and III·15, we get precisely the same form of equation as III·8 except that the values for K and A are obtained from equations III·13 and III·14 respectively.

Verbal Statement of the Formulas Showing their Meaning. In words all three formulas show that the original n-year annuity at a nominal rate, j, compounded m times per year and with a $1 annual rent, has been transformed into an equivalent annually paid, annually compounded annuity with a term m times as long, and interest rate $\frac{j}{m}$, or $\frac{1}{m}$th as great but with an annual rent that differs in the three cases as follows:

Case I. When $m = p$, the annual rent is $\frac{1}{m}$th of $1.

Case II. When $m < p$ and m is a factor of p, the annual rent is the amount to which installment payments of $\frac{1}{m}$th of $1, paid in k installments for one year, will amount at the end of the year at an effective rate, $\frac{j}{m}$ per cent.

[illegible]

EXERCISES

[illegible]

compounded quarterly?

[illegible]

Ans. $7881.65

[illegible]

term and frequency of payment and conversion by the amount of $1 for one installment period as expressed by $(1 + i)^{\frac{1}{p}}$ or by $\left(1 + \frac{j}{m}\right)^{\frac{m}{p}}$, according to whether the interest rate in the annuity is effective or nominal. For example, the present value of a *p*-paid annuity due with effective rate is:

$$A = R\mathrm{a}^{(p)}_{\overline{n}|i} = Ra^{(p)}_{\overline{n}|i}(1 + i)^{\frac{1}{p}} = Rs^{(p)}_{\overline{1}|i}(1 + i)^{\frac{1}{p}}a_{\overline{n}|i}$$

All the other cases of present value or amount of an annuity due are derived in the same manner.

It would seem that now and here might be a good time and place to tabulate those much-used and important formulas of annuities certain of all types in the following table which the student is to finish by filling in all blank spaces final formulas in the form best adapted for use with tables.

Kind of Annuity and Rate	Present value of $1	Amount of $1	Pres. value annuity of any rent	Amount of an annuity of any rent	Any dated value at any time
Simple annual annuity					
With effective rate					
With nominal rate					
Simple annuity due					
With effective rate					
With nominal rate					
Installment paid annuity					
With effective rate					
With nominal rate $m = p$					
$m < p$					
$m > p$					
Installment annuity due					
With effective rate					
With nominal rate $m = p$					
$m < p$					
$m > p$					

[illegible]

EXERCISES

[illegible]

8. Suppose, on the other hand, he had exchanged it for an annually paid one of the same annual rent; what loss would he have suffered?

9. State in words what the following symbols mean: $500 \times \frac{1}{s_{\overline{2}|.02}} \times s_{\overline{20}|.02}$.

§ 16. **Applying the Formulas.** Thus far our attention has been directed almost entirely to the derivation of a large group of formulas in such form that a maximum use may be made of tables. Although these many formulas contain about a dozen possible unknowns only one can be unknown in any one problem, and in practice the present value of the annuity, its amount, its annual rent, its term or its interest rate is the unknown whose value is sought in any particular problem.

Suppose, for instance, that all data is given except the amount. There are eight different formulas for the value of K among which choice must be made according to the kind of annuity given by the data: either annuity or annuity due; again either effective or nominal rate, and finally, if the interest rate is nominal, either $m \gtreqless p$, each with its own appropriate formula. If present value is the quantity involved instead of amount, a different set of formulas must be examined for the choice but the successive steps of the selection are made on the same basis.

The appropriate form of the fundamental formula having been chosen, judgment should be used as to whether it is expedient to substitute the known numerical values directly into this form. Often it is simpler to solve this equation for the unknown whose value is to be found before putting the known data into the formula. A saving of labor may be made whenever the "solved form" of the formula contains $s_{\overline{1}|}^{(p)}$ *in the denominator* by substituting $\frac{j_p}{i}$ for $\frac{1}{s_{\overline{1}|}^{(p)}}$ and then looking this up in Table X instead of using the former form and the tabular value in Table IX. This replaces a long division by a comparable multiplication. The situation calling for this substitution arises whenever the rent is the unknown to be determined.

When the unknown to be found is either the time or the rate the tabular symbol containing it is solved for and interpolation is resorted to for computing the solution. This brings only an approximate value for the result but in most cases a sufficiently [illegible] by a different method [illegible]

[illegible] found in Chapter VII.

[illegible] amount to $8400?

[illegible]

43.03

[illegible]

[illegible]

[illegible]

[illegible]

periods.

[illegible]

[illegible] paid annuity for 39 year[illegible]

[illegible]

[illegible]

[illegible]

Interest on [illegible]	Book value [illegible]
[illegible] of 19th year	[illegible] 7th month [illegible]
[illegible] of 19th year	[illegible] 8th month [illegible]
[illegible] of 19th year	[illegible] 9th month [illegible]

[illegible]

solved problem.

[illegible] IN ANNUITIES [illegible]

[illegible] $\left(1+\frac{j'}{4}\right)^4$ [illegible] $\left(1+\frac{j'}{2}\right)^2$ [illegible]

[illegible] $\left(1+\frac{j'}{2}\right)$ [illegible]

[illegible]

$$R\left[\left(1+\frac{j'}{4}\right)^{4\cdot n}-1\right]$$

[illegible] fundamental [illegible]

$$p\left[\left(1+\frac{j'}{4}\right)^{\frac{4}{p}}-1\right]$$

[illegible]

semiannual rate.

[illegible]

bank charging him?

[illegible]

$\frac{R}{12}$ [illegible] $\frac{R}{12}$ [illegible]

[illegible] $\frac{2400}{[illegible]}$ [illegible] R [illegible]

[illegible]

[illegible] per cent, [illegible]

[illegible]

[illegible]

[illegible] accuracy of [illegible]

[illegible]

to $200 in 7 years?

[illegible]

the formula:

[illegible]

[illegible]

$$\frac{2400 K \quad 24 R}{24 R}$$ [illegible]

[illegible]

[illegible] = 98.07[illegible] per cent, [illegible]

[illegible] of equal ratios, giving [illegible]

[illegible]

The accuracy of this approximation may be tested by its substitution into the fundamental form of the formula used for the solution of the problem (III·2 multiplied by $(1 + i)^{\frac{1}{p}}$ to make it apply to an annuity due), i.e.,

$$K = R \times \frac{(1 + i)^n - 1}{p[(1 + i)^{\frac{1}{p}} - 1]} (1 + i)^{\frac{1}{p}}$$

Substituting $n = 7$, $p = 12$, $R = 24$, $i = .0490341$ gives:

$$K = \frac{24[(1.0490341)^7 - 1]}{12[(1.0490341)^{\frac{1}{12}} - 1]} = (1.049031)^{\frac{1}{12}} \quad \text{or} \quad K = \$199.97073$$

The rate found therefore is a little too small. If we cut off one place, rounding in the usual manner, we shall make the rate still smaller, thereby decreasing still further the accuracy of our answer. But if we round *up*, using $i = .049035$, we get as a result of the test $K = \$199.9724$. Rounding *up* again makes $i = .04904$ and this gives as a test $K = \$199.9759$. Finally, using $i = .0491$ gives $K = \$200.0188$ instead of \$200 and we know that at last we have exceeded the exact rate desired.

From the problems worked in this section we must conclude that with formulas involving $s_{\overline{n}|}$ and $s_{\overline{p}|}$ the approximated rates found by the use of tables and interpolation have a tendency to be too small and should not be carried more than four decimal places, and in dropping decimal places rounding *up* should be resorted to even in the case of minor fractions.

EXERCISES

1. A monthly paid annuity with annual rent \$133.20 will accumulate to \$1000 at 6 per cent compounded monthly in what length of time?
Ans. 6 years, 3 months and \$4.04

2. If the monthly payment in Ex. 1 had been \$8.44, how long would it have taken? *Ans.* 7 years, 9 months, 23 days

3. If the annual rent in Ex. 1 had been \$85.92 paid monthly, how long would it have taken? *Ans.* 8 years, 10 months, 14 days

4. If a man consistently saves \$50 per month from the time he begins to earn a salary and deposits it in a savings bank paying 3 per cent semiannually compounded how long will it take him to accumulate \$30,000?

5. At what effective rate will monthly payments of \$10.37 accumulate to \$2000 in 11 years? *Ans.* $i = .067613$

6. At what rate will monthly payments of \$8.81 amount to \$2000 in 14 years?

[illegible]

MISCELLANEOUS EXERCISES

[illegible] 52 weeks? *Ans.* $106.12

[illegible] ability? *Ans.* $102

[illegible] tion? *Ans.* $106.04

[illegible]

Ans. $14,601.14

[illegible]

Ans. $353.43

[illegible]

MISCELLANEOUS EXERCISES

[illegible]

Ans. $6072.40

[illegible]

Ans. $2881.79

[illegible]

CHAPTER IV

PAYING DEBTS

[illegible]

in which T is the total annual sum, R is the annual rent of the sinking fund and F is the face of the debt.

It should be noted that the rate of interest, i', used in computing $s_{\overline{n|}}$ is not in general the same as i, the rate of interest paid by the debtor on the face of the debt.

The amount in the sinking fund at any time t years after the beginning of the term of indebtedness is the amount of an annuity for t years at rate i'.

Book value is used in both a narrow and in a broad sense in finance. In connection with sinking-fund operations it usually means the amount of principal left to be paid after each credit entry has been made and is spoken of as the book value of the principal at date of entry. In its more general use book value means the value on any given date of any item of a transaction the value of which varies from date to date or entry to entry on the account.

§ 18. **Amortization Method of Paying a Debt.** The method of amortization consists in paying the creditor directly a series of equal equispaced payments covering n years at the end of which the entire debt, principal and interest, is cancelled. These payments evidently constitute an annuity the present value of which equals the face of the debt.

If $A = R \times a_{\overline{n|}}$ is understood to represent the present value of an annuity of any type, then the annual rent of an annuity that will amortize a debt of F dollars, principal and interest, in n years is found from this formula solved for R:

$$R = \frac{F}{a_{\overline{n|}}} \qquad \text{[IV}\cdot\text{3]}$$

in which the general and vague symbol, $a_{\overline{n|}}$, is replaced by a definite form appropriate for the interest rate, method of compounding and frequency of payment specified in the contract for the payment of the debt. These appropriate forms of $a_{\overline{n|}}$ are to be found in formulas II·3, II·8, III·3, III·2 and III·14. In this method of paying a debt the amount left to be paid at any time is often required because the debtor may wish to settle before the expiration of the term of his loan by paying a lump sum. *This is found by finding the lump value of the annuity, which consists of the remaining payments,* dated at the date of the desired lump settlement.

A clear picture of many financial operations is obtained by the use of schedules showing the progress of the transaction from beginning to end. Amortization of a debt is a transaction of this type. A five-column schedule is made in this case. These columns are headed as follows:

1. The year.
2. Principal at the beginning of the year, sometimes called book value of debt.
3. Annual payment.
4. Part of annual payment paying the interest.
5. Part of the annual payment applied to amortizing the debt.

When the annual payment is such that a smaller last payment is needed to complete the amortization of the debt the schedule method is the one most commonly used for computing the progress of the loan.

However, when the time is long and the computation of a schedule would be tedious the equation of time can be used to compute the last payment. Suppose the first $n - 1$ equal payments are R each and the last payment to balance the account is R' and i is the effective interest rate paid in the problem. Then at the time of the loan $F = Ra_{\overline{n-1}|} + R'(1 + i)^{-n}$, whence solving for R' we get:

$$R' = F(1 + i)^n - R(1 + i)^n a_{\overline{n-1}|} = F(1 + i)^n - R(s_{\overline{n}|} - 1) \qquad [IV\cdot4]$$

§ 19. **Comparison of the Two Methods of Paying a Debt.** By the sinking-fund method of paying a debt the total yearly payment made by the debtor has been derived as:

$$T = R + Fi = \frac{F}{s_{\overline{n}|i'}} + Fi = F\left(\frac{1}{s_{\overline{n}|i'}} + i\right) \qquad [IV\cdot5]$$

By the amortization method the total yearly payment may be rewritten from equation IV·3, since the rent of the annuity equals the total yearly payment by the debtor.

$$T = \frac{F}{a_{\overline{n}|i}} \qquad [IV\cdot6]$$

If the rate, i', received by the debtor as interest on his deposits in the sinking fund that he has established in some bank for the

[illegible]

[illegible]

EXERCISES

[illegible] did he pay for the machine?

[illegible] company collecting? [illegible] from each month's milk check. How long did it take him to pay his debt if the bank charged him 5 per cent semiannually compounded interest?

Ans. 5 years, but only \$9.27 of last payment

5. A man owes two debts, each of \$5000, on which he is paying the same interest, 4 per cent effective, and both of which are due in 10 years, but one is to be paid directly to the creditor in equal annual installments paid at the end of each year and the other is to be paid by equal payments into a bank where it accumulates at 3 per cent semiannually compounded interest. When he gets them both paid he asks himself the question, "How much difference has there been between the two ways of paying these two debts?" Can you think of different ways of answering this question? Of the ways that you have considered for answering the question which seems to you to be best?

One answer: \$19.25 in each annual payment;
Second answer: difference in present values is \$160.68

6. Just after making the fourth payment into the sinking fund for the payment of a debt of \$800 which he had agreed to pay in 7 years with interest at 5 per cent, the debtor sees his way to paying the entire debt at once. His sinking fund has been accumulating at 4 per cent effective. How much is there needed to complete his sinking fund for the immediate payment of the debt?

7. A debt of \$900 together with interest at 3 per cent is to be paid by accumulating both the principal and the interest in a sinking fund that accumulates at 4 per cent, both rates being effective. Make a schedule to show the progress of the fund if the debt is to be met when due 4 years from now.

Ans. $R = \$238.54$

8. Suppose the debt in Ex. 7 were to be met in the usual manner by paying the interest directly to the creditor annually and accumulating only the debt. Make a schedule that shows all essentials of the progress of this operation.

9. A debtor who owes a debt of \$1000 due in 6 years with interest at 6 per cent has been making monthly payments into a sinking fund accumulating at 3 per cent compounded semiannually for 4 years when his creditor, having immediate need for cash, offers to remit his year's interest just due and being paid, if the debtor will pay his debt at once. How much more money must the debtor raise in order to pay now? What effect would this bargain have upon the interest rate actually paid by the borrower for the use of the money?

10. If the original contract had been carried out as made in Ex. 9 would the borrower be paying the 6 per cent effective that he appears to on the first glance at the problem? If not, prepare a solution and argument to defend your conclusion as to the rate he is paying.

MISCELLANEOUS EXERCISES

1. To repay a borrowing of K dollars borrowed at rate r per cent a man agrees to pay $2Kr$ dollars per year until the debt is fully paid. Construct a schedule for the repayment of the loan covering the first 4 years.

2. A man buys a house paying \$12,000, one-half the cost of the house, in cash. If he pays thereafter \$3000 per year covering principal and interest and

[illegible]

Ans. $16696.80

[illegible]

Ans. Six years; $898.57

[illegible]

Ans. $169.68

[illegible]

Ans. $625.30

[illegible]

SUPPLEMENTARY EXERCISES ON CHAPTER IV

1. In order to save money to buy an automobile costing \$2765, a man deposits at the end of every 6 months a certain sum in a savings bank paying 3 per cent nominal convertible semiannually. He wishes to buy the automobile at the end of 3 years. What is the semiannual deposit? *Ans.* \$443.85

2. If the man in Ex. 1 borrowed the money to buy the auto now and arranged to pay the debt, principal and interest, in 6 equal semiannual payments beginning at the end of the first half-year, what would be the semiannual payment if the interest rate was 6 per cent nominal convertible semiannually? *Ans.* \$510.41

3. If in Ex. 2 the man used the savings bank of Ex. 1 to accumulate a sinking fund, what sum must he have in hand at the end of each half-year to pay interest and to extinguish the debt in 3 years? *Ans.* \$526.80

4. A man borrows P dollars at rate i and arranges to pay $2Pi$ dollars per year until the debt is paid. Prepare a schedule showing the principal and interest paid each year for the first 4 years.

5. A man borrows \$1000 at 5 per cent and arranges to pay twice the amount of the original interest each year until the debt is extinguished. Construct a complete schedule of repayment.

6. A man buys a house for \$12,000 paying one-half down. He arranges to pay \$1500 per year, principal and interest, on the remaining amount at 6 per cent until the debt is paid. How many payments of \$1500 are made and what is the amount of the final payment? *Ans.* Four; \$426.29

7. At the end of 2 years what was the owner's equity in the house of Ex. 6? *Ans.* \$8348.40

8. A county borrows \$25,000 to build a bridge. The payment is to be made by amortization of the principal for 15 years at 6 per cent. At the end of the tenth year what part of the debt remained unpaid? *Ans.* \$10,842.90

9. A sidewalk tax of \$174.83 is to be paid in 5 equal annual installments with interest at 5 per cent. Construct a schedule of the payments showing for each year the interest payment, the book value of the indebtedness, and the book value of credit on the tax to taxpayer.

10. A firm operating on a 13-month year arranges to pay a debt of \$7000, principal and interest by equal monthly payments for 5 years. What is the monthly payment if the interest rate is $5\frac{1}{2}$ per cent? *Ans.* \$123

11. An automobile truck costing \$2000 and lasting 5 years (at the end of which time it has a secondhand value of \$350) is to be replaced by means of a sinking fund accumulated at 4 per cent from annual payments during 5 years. At the beginning of the fourth year the truck is destroyed by an accident with a junk value of \$50. What amount must be added to the sinking fund and junk value to replace the truck destroyed? *Ans.* \$999.05

12. A fraternity chapter borrows \$65,000 at 7 per cent to build a house. A sinking fund can be built up at 5 per cent. What amount must be raised yearly to pay this debt if arrangements are made to extend the payments over 30 years? *Ans.* \$5528.34

13. A debt, P, is to be paid, principal and interest, in n years by equal pay-

[illegible]

[illegible]

$\frac{P}{p} \times j_{(p)} \times \frac{1}{a_{\overline{n}|}}$ payment is

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] is paid. [illegible] the debt [illegible]

[illegible]

[illegible] formula for [illegible]

[illegible]

[illegible]

[illegible]

[illegible] debt remained [illegible]

Ans. \$1666.40

CHAPTER V

DEPRECIATION

When capital is invested in any kind of business enterprise it is sure to change in value as time goes on. Some of it may be invested in the machinery and tools of the enterprise which wear out with use or become less valuable through the introduction by competitors of better and more efficient machinery; some of it may be spent for the buildings to house the industry and some to keep them in repair if they are not to deteriorate and soon become worthless; some of it may be invested in raw materials that are perishable—in short there must be constant care and effort exercised to keep the original capital intact. Some of this decrease in value is cared for by current replacements and repairs, but things of longer life and smaller need for repairs from month to month cannot be replaced in this way without causing sudden big drains upon current assets that may seriously embarrass the management.

Such large-scale, long-time decrease in capital must be provided for by an appropriate long-term method of accumulating a sum equal to the depreciation of the capital, a method which as nearly as possible parallels in time the shrinkage in the value of the capital or, rather, in the value of the objects in which the capital is invested. All of the various methods that have been devised for caring for depreciation make use of certain fundamental quantities in common. We may therefore define these general terms once and for all before considering the various methods individually.

The changing value of the depreciating article is called its *book value* and the symbol B_t is used to symbolize the book value at the end of t years from purchase. The book value starts as C, the cost, and finishes at the end of n years, the useful life of the article, as S, the *scrap value.* D_t is the depreciation charge of the tth year. The difference between the original value or cost and the book value at the end of t years is $T_t = \sum_0^t D_t$ the *accumulation of depreciation*

[illegible]

$$T_t = C - B_t = \sum_0^t D_t \qquad [V\cdot 1]$$

$$W = C - S \qquad [V\cdot 2]$$

$$W_t = W - T_t \qquad [V\cdot 3]$$

$$c = \frac{W_t}{W} \qquad [V\cdot 4]$$

[illegible]

[illegible]

$$D = \frac{W}{n} = \frac{C - S}{n} \qquad [V\cdot 5]$$

For purposes of comparison of the several different methods of caring for depreciation, each of the five general methods will be illustrated by the same problem solved by the formulas, put into schedule form and pictured by a graph. The problem used for all five methods is as follows:

PROBLEM. An article costing \$1200 has an estimated life in use of 6 years and a scrap value of \$400 at the end of that time. Show in schedule form and picture by graph the progress of its depreciation.

SOLUTION. It will be noted that the schedules are not always put in the order in which the computations are made. In solving any problem by the straight-line method, for example, the items in column 3 are computed first by the use of equation V·5, letting t range from 0 to 6 inclusive. Equation V·1 is then used for computing column 4 and equation V·7 for computing the items of column 5. Finally, the items in column 6 are computed by the use of equation V·4. There are several checks by which you can tell whether your computations are correct. See if you can discover them by a study of the schedule, the formulas or both.

SCHEDULE SHOWING THE PROGRESS OF DEPRECIATION AND THE UP-BUILDING OF FUNDS BY THE STRAIGHT-LINE METHOD

1 End of year	2 Book value	3 Annual depreciation charges	4 Total depreciation charges	5 Remaining wearing value	6 Condition per cent
0	\$1200.00	....	....	\$800.00	100%
1	1066.67	\$133.33⅓	\$133.33	666.67	83⅓%
2	933.34	"	266.67	533.33	66⅔%
3	800.00	"	400.00	400.00	50%
4	666.67	"	533.33	266.67	33⅓%
5	533.34	"	666.67	133.33	16⅔%
6	400.00	"	800.00	0	0

§ 21. **The Constant Percentage Method.** Some things, such as a farmer's truck or his pleasure car, lose their value in use very rapidly at first and more slowly as time goes on. The straight-line method of caring for this loss of value would leave the owner almost entirely unprotected in case he were obliged to transfer his property during the first few years of his ownership. When depreciation is to be financed on anything that depreciates in this

[illegible]

[illegible]

centage of shrinkage.

[illegible]

$$B_t = Cb^t \quad [V\cdot 8]$$

[illegible]

$$S = Cb^n \quad [V\cdot 9]$$

[illegible] found by solving V·9

$$b = 1 - r = \sqrt[n]{\frac{S}{C}} \quad [V\cdot 10]$$

$$r = 1 - \sqrt[n]{\frac{S}{C}} \quad [V\cdot 11]$$

[illegible] problem [illegible]

[illegible]

METHOD

so that the total annual increase in the depreciation fund is composed of the sum of this constant deposit, D, for the current year, and the current year's interest upon the amount already so accumulated. It is this sum of D and the current interest on the accumulated previous deposits of D that constitute the real yearly provision for depreciation. Therefore,

$$D_t = D + Dis_{\overline{t-1}|i} = D\,(1+i)^{t-1} = \frac{W}{s_{\overline{n}|i}}\,(1+i)^{t-1} \qquad [\text{V}\cdot 13]$$

Also the items in T_t are the sum of the previous and current items of D_t from which it becomes evident that:

$$T_t = Ds_{\overline{t}|i} \qquad [\text{V}\cdot 14]$$

The illustration already solved by two methods cannot be done by this method until one additional piece of data is given, the rate paid in the sinking fund. Let us restate the problem for this method as follows:

Problem. An article costing $1200 has an estimated service life of 6 years and a scrap value of $400 at the end of that time. Show in schedule form and picture by graph the progress of its depreciation by the sinking-fund method with the sinking fund accumulating at 5 per cent.

Solution. The items of column 3 are computed from equation V·13 by taking $i = 5$ per cent and t varying from 1 to 6. The items of column 4 are computed from equation V·14 and the use of Table III with t varying from 1 to 6, or it may be found by adding previous and current items of

Schedule Showing the Progress of Depreciation and the Up-Building of the Funds to Care for It by the Sinking-Fund Method

1 End of year	2 Book value	3 Annual addition to sinking fund	4 Total accumulation in sinking fund	5 Remaining wearing value	6 Condition per cent
0	$1200.00	....	....	$800.00	100%
1	1082.39	$117.614	$117.614	682.39	85.3%
2	958.89	123.50	241.11	558.89	69.86%
3	829.22	129.67	370.78	429.22	53.65%
4	693.07	136.15	506.93	293.07	36.63%
5	550.11	142.96	649.89	150.11	18.75%
6	400.00	150.11	800.00	0	0

[illegible] computations.

[illegible]

[illegible] lated.

[illegible] which follows:

$$I_t = B_t + i \qquad [V \cdot 15]$$

[illegible]

Column 8 is formed from the sums of corresponding items of columns 3 and 7.

This method received its name from the fact that its total annual payment, including the interest allowances on sinking fund and depreciating invested capital, is constant and is consequently like the annual rent of an annuity. It is employed occasionally both when the depreciation funds are invested outside the business in interest-yielding securities and when they are kept employed in the business, and these funds are merely a matter of bookkeeping.

§ 24. **The Compound-Interest Method.** This second modification of the sinking-fund method was devised by the American Society of Civil Engineers in 1927 to be used in the valuation of public utilities, such as railroads and electric systems. By this method the funds allotted to care for depreciation are kept employed in the enterprise. The depreciation sinking fund is therefore a kind of bookkeeping amortization of the depreciation by the annual transfer to the depreciation account not only of the fixed annual sum found as the annual rent of the sinking fund as computed by the method given in § 22, V·12, but also of the annual interest at " sinking-fund rate " on the accumulation in the sinking-fund account and of interest at a fair rate for this kind of investment, i (different from the sinking fund rate, i') on the remaining book value of the investment.

The first sinking-fund charge, by § 22, V·12, is seen to be $D = W \times \frac{1}{s_{\overline{n}|i'}}$.

The sinking-fund charge of any year, t, is, by V·13, $D_t = W \times \frac{1}{s_{\overline{n}|i'}} \times (1 + i')^{t-1}$.

The total accumulation on the books in the depreciation account is, by V·14, $T_t = Ds_{\overline{t}|i'}$.

So far the computations are identical with those of the sinking-fund method and the first six columns of the two methods will be exactly the same. In the compound-interest method, however, there is a seventh column the items of which are the compound-interest charges on the depreciated book value which the article has at the beginning of that year computed at a rate, i, different from that used for the sinking-fund accumulations.

It is therefore necessary to first compute the book value at the

[illegible]

[illegible]

[illegible]

[illegible] the progress of the fund.

[illegible]

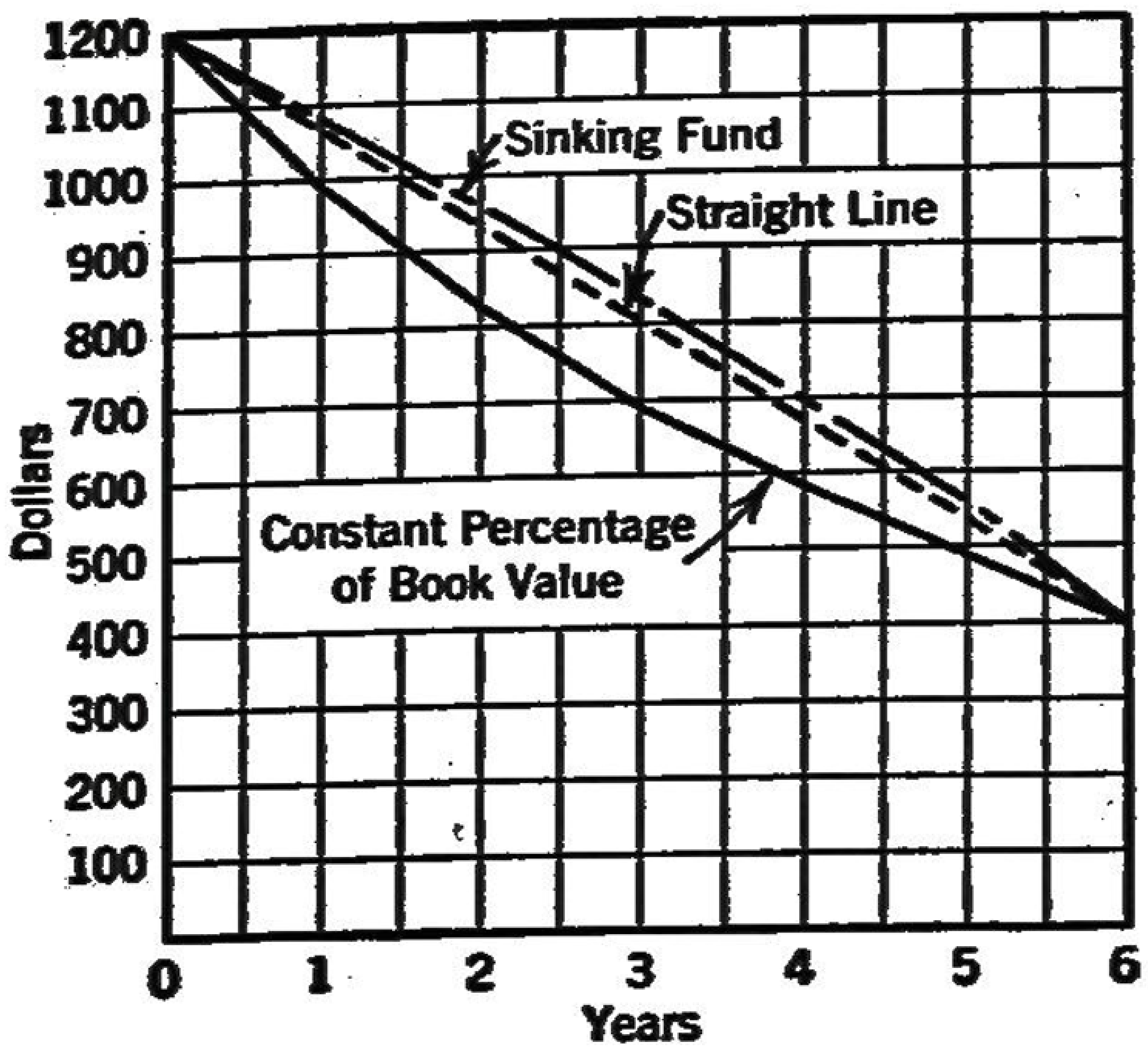


FIG. 9. Chart comparing book values in straight-line, constant percentage of book value, and sinking-fund methods of depreciation.

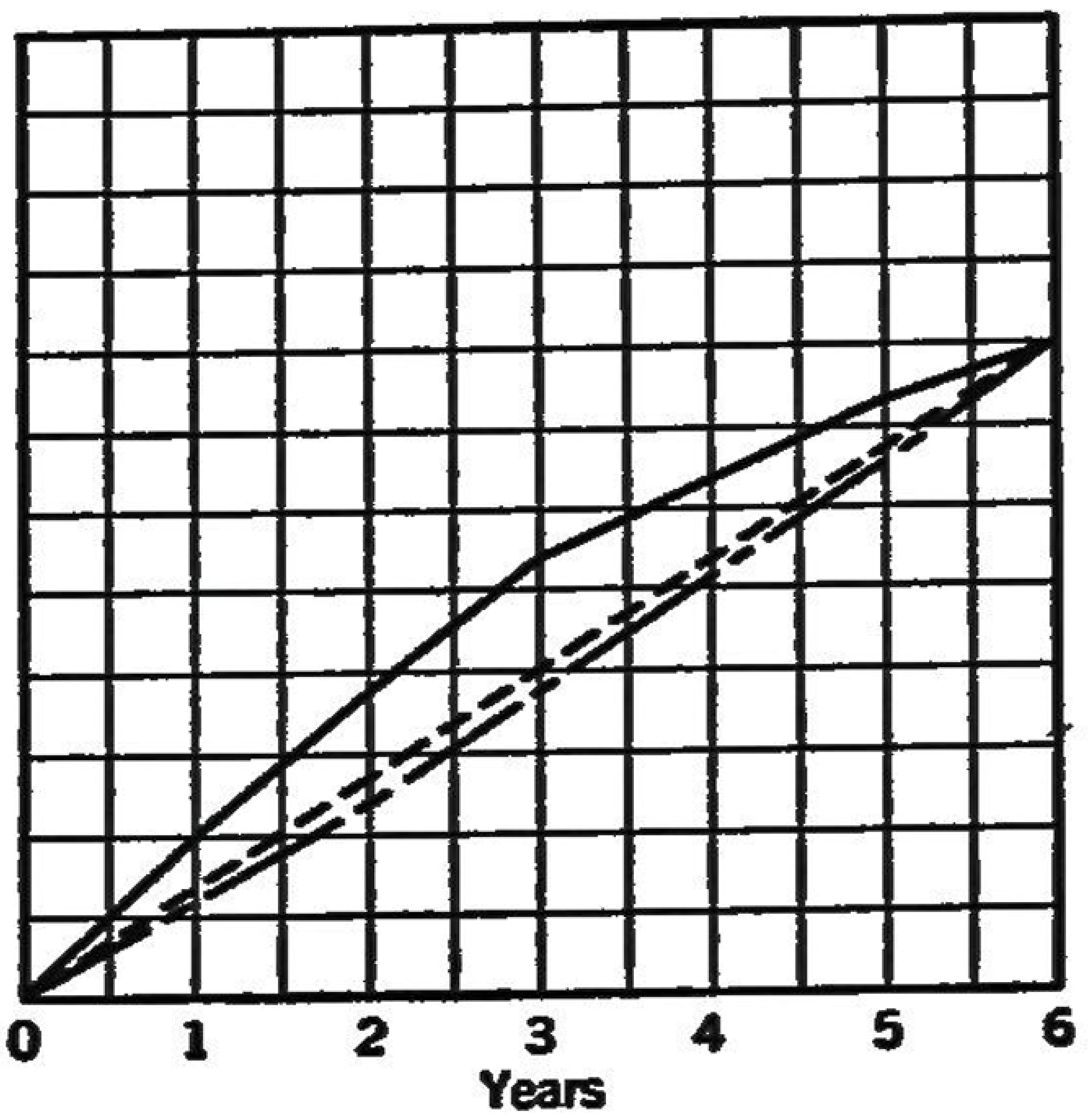


FIG. 10. Chart comparing accumulation of depreciation funds by the same three methods.

EXERCISES

1. A suburbanite buys a car for $600 with which to travel to and from his work in the city. At the end of 6 years he concluded to sell and found that he could get only $50 for it. Had he provided a depreciation fund with which to buy another car, show a schedule of it by the straight-line method. *Ans.* D = $91.67

2. Show a schedule of the progress of his fund had he used the constant-percentage method of funding his depreciation. *Ans.* $r = .3391$

3. Make a schedule of the sinking-fund method of caring for his problem if he can get 4 per cent on his sinking fund. *Ans.* D = $82.92

4. Change your schedule as much as necessary to turn his account into the annuity method. What does the annuity method do for him that the sinking-fund method did not?

5. Use the compound-interest method and compare it with the two preceding methods as to just what it provides for that they do not. Use 6 per cent on investment.

6. Which of the above methods would in your opinion most closely follow the facts in the above depreciation problem? State your reasons for your choice.

7. Had the above car been involved in an accident just as he decided to sell it which method of depreciation would have become absurd and unworkable?

8. Plot all of the above depreciation methods on a single graph on a large scale.

§ 25. The Composite Life of a Plant. These five methods of caring for the deterioration of any machine, implement, or equip-

[illegible]

considered.

[illegible]

scrapped.

[illegible]

of the plant, then

[illegible]

composite life of the plant. In this case, therefore, the n is found by interpolation with Table III from the formula:

$$s_{\overline{n}|i} = \frac{W}{D} \qquad [V \cdot 19]$$

It is understood that W is the sum of all the separate wearing values of the parts of the plant and D is the sum of the separate depreciation charges each found by the sinking-fund formula.

Instead of interpolation from the table the sinking-fund formula may be written in the form $\frac{(1+i)^n - 1}{i} = \frac{W}{D}$ and solved for n, giving:

$$n = \frac{\log (Wi + D) - \log D}{\log (1 + i)} \qquad [V \cdot 20]$$

and the value of n found by logarithms.

EXERCISES

1. A plant is composed of the following parts having the values and life shown in the accompanying table. Find the composite life of the plant using (a) the straight-line method and (b) the sinking-fund method, at 5 per cent.

Part of plant	Value	Estimated life	Scrap value	Depreciation charges	
				Straight line	Sinking fund
Buildings	$160,000	50 yrs.	$6000		
Machinery	75,000	16 yrs.	8000		
Furnishings	5,000	12 yrs.	500		
Fixtures	2,000	25 yrs.	400		
Tools	8,000	4 yrs.	500		

2. Having found the composite life of the above plant by the straight-line method, find the remaining wearing value in those items which have any wearing value left in them at the end of the composite life of the plant.

3. Do as directed in Ex. 2 except in this case use the sinking-fund method.

4. If T_t represents accumulation or sum of depreciation charges at the end of t years find an expression for T_t true in general in the constant-percentage method. Also for the general annual depreciation charge, D_t. Test the value of your result by use of the problem solved in the text.

5. Find the condition per cent of each part of the plant at the end of the composite life of the plant in Ex. 1, method (b).

6. Derive formulas to care for all essential quantities to be found in the annuity method of caring for depreciation.

7. A turbine-driven electric generator costing $85,000 is to be installed by a big power and light company. The company earns 12 per cent on money invested in the plant. The local bank pays 4 per cent on savings deposits. The generator is estimated to have a service life of 30 years and a turn-in value of $15,000. Make out the first 4 years of the depreciation schedule for the generator by each of the five methods of caring for depreciation.

§ 26. **Depreciation of Oil, Mining and Timber Lands.** In this class of investments the value of the property lies almost entirely in the oil, mineral or lumber. When this product has been removed the value of the investment will have shrunk to practically zero. Any sound system of financing, therefore, must provide a reasonable annual return on the capital invested and a sufficiently large yearly payment into a sinking fund safely invested so that by the end of the profitable life of the project the accumulation from this so-called " Redemption Fund " will restore completely the original capital invested.

The oil expert, the mining engineer or the timber cruiser first furnishes an expert estimate based on investigation of the property concerned. His estimate usually evaluates not only the total value of the property but its average yearly output for an estimated number of years. From him also or from other experienced sources the gross and net cost per unit for delivery of the product to a known market is estimated when delivered at a certain predetermined rate per year. The rate at which a sinking fund can be accumulated and the rate that an investment of such risk should earn are the two other items needed before the promoters of such an investment can make their computations to determine what capital should be the maximum set for developing the enterprise.

Let P = the estimated potential net value of the property, or present value
R = the net annual income from the property (also estimated)
I = the amount of yearly income that P should earn when invested in such a risky enterprise, $I = P \times i$
S = the annual payment into the redemption fund accumulating at rate i', $S = \frac{P}{s_{\overline{n|}i'}}$

It is evident that $R = I + S = P \times i + \frac{P}{s_{\overline{n}|i'}} = P\left(i + \frac{1}{s_{\overline{n}|i'}}\right)$, which solves into:

$$P = \frac{R}{i + \frac{1}{s_{\overline{n}|i'}}} \qquad [\text{V}\cdot 21]$$

EXERCISES

1. Should the above class of investments lose the element of risk to such an extent as to become like ordinary investments and the rate i' become equal to i, show that equation V·21 reduces to the ordinary formula for the cost of an annuity. Does this mean that one gets i on his investment and gets it all back by the end of its term when he buys an ordinary annuity?

2. The net annual revenue from a lumbering project is \$10,000 for the next 10 years. If the risks involved make 10 per cent a fair return on investment and if 4 per cent is earned on the redemption fund, what sum could the contractor afford to invest? *Ans.* \$54,558.08

3. In the Venice, California, oil field, a certain group of the wells averaged to produce 100 barrels per day for 5 years and the oil sold at the well for \$1 a barrel. It cost an average of \$40 per foot to bring them into production and a redemption fund was established at 4 per cent. What rate of income was earned on the investment? The average depth of the wells was 2000 feet.

4. In Ohio the coal lies in a fairly uniform layer 4 to 6 feet under the surface of the level land. A coal company bought 5000 acres of this land at \$125 per acre. It took 4 years to organize the plant and get into production, during which time a yearly tax of \$5000 per year was paid on the land. The engineer's estimate gave the life of the mine as 50 years at 5000 tons per acre. The entire equipment cost \$500,000 and will have a composite life of 15 years. The value of the land when exhausted will be \$10 per acre. The annual cost of labor and power will be \$542,000 and for taxes, repairs and administration will be \$200,000 per year. Coal could be sold at the mine for \$2.25 per ton. Find out the total investment in the mine at the end of 10 years after purchase. Also find the value of the mine at that time. Assume interest rate of 5 per cent effective, that all sales and expenses occur at the ends of the years to simplify the accounts and that production is annually constant.

5. A man patents an invention and fails to get capital to manufacture. He finally sells it for \$100,000 when it has 10 more years to run. What net annual profit must the buyer make if the plant for manufacture costs \$5000? He can get 4 per cent on a sinking fund and plans to get 10 per cent return on his investment. *Ans.* \$19,245.59

6. What price can a company afford to pay for a copper mine that will promise \$30,000 per year for 18 years if 12 per cent is to be earned on investment and the sinking fund can be invested at 4 per cent? *Ans.* \$188,687.20

7. A logging company holds timber lands that promise a net income of \$30,000 per year for the next 20 years. If the company must make 10 per cent

on its investment and can earn 4 per cent on its sinking-fund investment what is the value of the investment? *Ans.* $224,581.49

8. A timber tract can be bought for $1,000,000. It is estimated that the tract will yield an annual net income of $200,000 for 10 years and that the cleared land can then be sold for $20,000. The logging company wishes to make 10 per cent on the investment and can earn 4 per cent on redemption funds. Is it a good buy? How much margin does it give them for inaccurate estimates? *Ans.* (a) $1,091,161.65, (b) $111,161.65

§ 27. **Choice in Buying or Replacing an Article, Based on Depreciation.** When buying an article, implement or machine for some permanent use or operation the buyer is confronted by an embarrassment of choices. The relative annual depreciation costs of the articles under consideration furnish a scientific basis for a decision under such circumstances. Two important cases arise: (1) when the things are used exclusively or mainly for the production of a definite average number of units of production of some kind, such as a machine for making wood screws known to produce a fixed number of gross per day; (2) implements or articles *not* used for the direct fabrication of an average countable number of units of a given product.

CASE 1. This case is dealt with by what is known as the *unit cost* method. The excuse for putting this method under depreciation is that it is frequently used in determining the time when paying the depreciation and repair charges on the old machine now in service makes it more expensive to run than it would be to buy a new one to replace it.

Let P = the cost or present value of the new machine
D = the annual payment that will amortize its cost, principal and interest on the remaining book value, in N years
O = annual operating expense exclusive of repairs and depreciation
R = average annual cost of repairs on it
N = the estimated useful life of the machine
i = the open-market rate of interest
U = the number of units of product fabricated by it per year

Let the small letters, p, d, o, r, n and u, represent the same quantities as they apply to the old machine or the other new machine

[illegible]

[illegible]

$$\frac{R + O + D}{U}$$

[illegible]

$$\frac{r + o + d}{u}$$

[illegible]

$$\frac{r + o}{u} \quad \frac{R + O + D}{U}$$

[illegible] machine, giving:

$$\left(\frac{R + O + D}{U}\right)$$ in which

Substituting and solving [illegible]

$$\frac{R + O + D}{U}$$

When the number of [illegible] is the same [illegible] two machines [illegible] equation reduces [illegible]

$$O + D \left[\sqrt{(\ldots)}\right]$$

When the sum of [illegible] also equal [illegible]

$$p = Da_{\overline{n}|i}$$

$$\frac{Pa_{\overline{n}|i}}{[\sqrt{\ldots} \cdot 24]}$$

[illegible]

[illegible] the one first considered [illegible]

[illegible]

[illegible] in their respective lives [illegible]

$$\frac{C+y}{a_{\overline{n}+i}} = \frac{C}{a_{\overline{n}|i}}$$

Solving for y, [illegible]

giving as the final form

$$y = \frac{Ca_{\overline{n}|i}}{s_{\overline{n}|i}} \qquad [IV \cdot 25]$$

[illegible] same one, as can be shown [illegible]

EXERCISES

[illegible]

MISCELLANEOUS EXERCISES

[illegible] should occur [illegible] the replacement [illegible]

[illegible] old machine will make [illegible]

[illegible] old machine valueless.

Ans. 278 units

10. Two new machines [illegible]

[illegible] per cent?

Ans. $3053.50

[illegible] What [illegible] for both [illegible]

[illegible] prices of the two machines [illegible] second better by $227.47

MISCELLANEOUS EXERCISES

1. The trucks of a truck [illegible] company [illegible] per cent [illegible] earning?

Ans. 4.147 per cent

2. [illegible]

life of a tie be extended by chemical treatment that costs $1.40 per tie, if interest is at 4 per cent? *Ans.* 8 years

5. Ordinary cedar posts can be bought for 40 cents each and will last 18 years. If interest is at 4 per cent how much can a farmer spend upon chemical treatment that will double the life of his fence posts? *Ans.* 20 cents

6. Find the composite life of a plant (1) by the straight-line method; (2) by the sinking-fund method at 3 per cent:

	Value new	Scrap value	Estimated life	Wearing value	Depreciation charge
A.	$18,000	$3,000	15 years		
B.	50,000	10,000	40 "		
C.	30,000	10,000	40 "		

Ans. (1) 30 years; (2) 29.67 years

7. Find the constant per cent by which the value of a machine is decreased if its cost is $12,000, its scrap value $200 and its estimated life 15 years. *Ans.* 11.26 per cent

8. What is the capitalized cost of a truck that costs $18,000, lasts 6 years and has a final value of $100? Interest is at 6 per cent effective. *Ans.* $6798.59

9. If the machine in Ex. 7 had depreciated in such a way that the straight line method could have been used what would have been the annual depreciation charge? *Ans.* $66.67

10. Had the sinking-fund method of depreciation been used for the machine in Ex. 7 and a 3 per cent interest rate employed what would the ninth annual depreciation charge have been?

11. A shallow coal mine is estimated to give a net income of $30,000 annually for 8 years. The income rate allowed is 8 per cent and the rate on the redemption fund is 5 per cent. What is the estimated value of the mine? *Ans.* $162,411.87

12. One of the old college buildings has been financed as follows: cost, $300,000, yearly repairs (paid at the end of each year), $1000, extensive renewals (paid at the end of each 10 years), $12,000, sale price at the end of 40 years (before renewals), $10,000. What annual sum should have been set aside to care for repairs, renewals and the recovery of depreciation on capital?

13. Do Ex. 10 by the annuity method making out the first 3 years and the tenth year of the depreciation schedule.

CHAPTER VI

[illegible]

[illegible]

[illegible] payments.

[illegible] The place of payment.

[illegible]

[illegible]

ration creates a *bond issue* of $20,000,000. Assuming that the corporation deems it necessary to secure this huge loan in order to make it satisfactory to the many potential lenders we shall next find a mortgage being placed upon the properties of the corporation.

So far nothing is known of who will be the lenders or to what amounts they will participate in the loan. Under these circumstances the corporation cannot very well make out thousands of parti-mortgages for as yet unidentified lenders. It therefore creates one mortgage and conveys that mortgage to a trustee who will act in the interests of the many lenders. But instead of making out one promissory note as would the individual, the corporation has *bonds* printed in denominations of, say, $500, $1000, $5000 and $10,000 to an aggregate of $20,000,000. These bonds are really glorified promissory notes, fundamentally as simple as a note executed by an individual.

Each bond will state the amount owed the bearer or owner of the bond and that amount will be either $500, $1000, $5000 or $10,000. It will also state the rate of interest to be paid, the dates on which it is to be paid, where the funds for interest payment may be obtained by the lender, the date upon which the bond will be paid and where it will be paid. It is quite possible that the corporation will want to pay off all or part of the loan before the agreed-upon maturity date. In such case it makes provisions for anticipating payment by providing one or a series of " call dates " before the maturity date. Furthermore the bond will state all the other conditions of the loan, including the mortgage security held by the trustee. Inasmuch as the corporation does not know beforehand who is going to own its bonds, it is evident that it must provide a convenient way for those who will own its bonds to obtain the interest payments. This is accomplished by a very simple device.

To be concrete let us suppose that the $20,000,000 already referred to was " floated " (put on the market) January 1, 1939, to be paid or redeemed January 1, 1954, with interest paid July 1 and January 1 beginning July 1, 1939, and ending on the maturity date, January 1, 1954, 15 years after issue. The interest is at 4 per cent paid semiannually, hence each interest payment on a $1000 bond is $20. This makes 30 interest payments in all. The bond itself is designed to carry 30 small rectangles of paper, [illegible] *coupon bond.*

[illegible] creditors.

[illegible]

ever beyond the reputation of the management for meeting its obligations. Such a bond issue would be called *debenture bonds.*

The bonds actually issued by our hypothetical corporation, which we will call the Royal Air Corporation, would be known in the market as Royal Air 1st Mortgage 4's, 1954.

The account of how to contract a debt by the bonding method would not be complete without a moment's consideration of the mechanism by which the Royal Air Corporation actually sold its bonds to the investing public—individuals, banks, insurance companies, trust companies and other institutions having funds to invest. Under existing federal laws the corporation first had to register its bond issue with the Securities Exchange Commission (SEC), submitting all pertinent facts concerning the issue. A few of these facts would be: a history of the corporation, a complete description of the properties upon which the mortgage was placed, a discussion of the policy of the management in the handling of depreciation, a discussion of its capitalization, earnings, contingent liabilities, relationship of the corporation to the trustee and to the *investment bankers* who will actually sell the bonds to the investing public. This, of course, implies that all arrangements with the investment banker or bankers had been made prior to the submission of the issue to the SEC for registry. If and when the SEC approves the issue and places it in registry the investment bankers can go ahead with their plans for the sale of the bonds to the general public.

Among the prearrangements between the corporation and the investment banker is the sale to the banker of the entire issue at a certain fixed price, usually considerably below the par value of the bonds or at least considerably below the price which they are expected to bring when offered to the public. This ends the transaction for the corporation as far as the sale of the issue is concerned and shifts the responsibility to the shoulders of the investment banker.

But he is a specialist with expert knowledge as to the quality of this type of investment. He is in touch with the investment markets and knows such details as what combination of interest rate and life of bond will be acceptable to the investing public and how to reach those brokers, dealers and investment bankers who have extensive acquaintance with all types of investors.

[illegible]

[illegible] bonded debt is floated [illegible] invested [illegible] more understandable.

The [illegible] Bonds are seldom [illegible]

[illegible]

10 per cent.

[illegible]

computations.

[illegible]

Rubber	6½	[illegible]	[illegible]	'21	97½	100½	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Co.	[illegible]	par [illegible]	[illegible]	'20	100½	103½	95½	101½	100½	4.10	4.80
Co.	[illegible]	[illegible] '40	[illegible]	'35	100	105¼	105	100	100½	[illegible]	[illegible]
SF	75	[illegible] '37, 2 '47	JaS	'25	99½	110	107½	111	110½	6.05	6.20
Co.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	83		[illegible]	[illegible]
	5¾	NC '42 '52	[illegible]	'32	100½	100½	[illegible]	105½	[illegible]	5.10	[illegible]
R. R.	4½ Int.	NC [illegible] Exempt.	[illegible]	'24 Floating	[illegible]	[illegible]	90 Low	92 High & Low	[illegible]	[illegible] 5/15	'30 [illegible]
Co.	6 rate	[illegible] rate and date	[illegible] dates	[illegible] rate and price	[illegible]	[illegible]	[illegible]	[illegible] 5/15	'30 [illegible]	[illegible] Yield to	[illegible] Stock
Co.	4¾	[illegible] '51, 6 '52	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	100¼	[illegible] maturity	[illegible] yield
	[illegible]	par '55	jA	'25	[illegible]	101¼	98¼	100¼	[illegible]	[illegible]	[illegible]
	6½	NC '31	jD	'21	95	105	103½	102½	103½	5.95	6.12
Rubber	6½	[illegible]	[illegible]	'21	97½	100½	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Co.	[illegible]	par [illegible]	[illegible]	'20	100½	103½	95½	101½	100½	4.10	4.80
Co.	[illegible]	[illegible] '40	[illegible]	'35	100	105¼	105	100	100½	[illegible]	[illegible]
SF	75	[illegible] '37, 2 '47	JaS	'25	99½	110	107½	111	110½	6.05	6.20
Co.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	83		[illegible]	[illegible]
	5¾	NC '42 '52	[illegible]	'32	100½	100½	[illegible]	105½	[illegible]	5.10	[illegible]
R. R.	4½ Int.	NC [illegible] Exempt.	[illegible]	'24 Floating	[illegible]	[illegible]	90 Low	92 High & Low	[illegible]	[illegible] 5/15	'30 [illegible]
Co.	6 rate	[illegible] rate and date	[illegible] dates	[illegible] rate and price	[illegible]	[illegible]	[illegible]	[illegible] 5/15	'30 [illegible]	[illegible] Yield to	[illegible] Stock
Co.	4¾	[illegible] '51, 6 '52	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	100¼	[illegible] maturity	[illegible] yield
	[illegible]	par '55	jA	'25	[illegible]	101¼	98¼	100¼	[illegible]	[illegible]	[illegible]
	6½	NC '31	jD	'21	95	105	103½	102½	103½	5.95	6.12
Rubber	6½	[illegible]	[illegible]	'21	97½	100½	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Co.	[illegible]	par [illegible]	[illegible]	'20	100½	103½	95½	101½	100½	4.10	4.80
Co.	[illegible]	[illegible] '40	[illegible]	'35	100	105¼	105	100	100½	[illegible]	[illegible]
SF	75	[illegible] '37, 2 '47	JaS	'25	99½	110	107½	111	110½	6.05	6.20
Co.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	83		[illegible]	[illegible]
	5¾	NC '42 '52	[illegible]	'32	100½	100½	[illegible]	105½	[illegible]	5.10	[illegible]
R. R.	4½ Int.	NC [illegible] Exempt.	[illegible]	'24 Floating	[illegible]	[illegible]	90 Low	92 High & Low	[illegible]	[illegible] 5/15	'30 [illegible]
Co.	6 rate	[illegible] rate and date	[illegible] dates	[illegible] rate and price	[illegible]	[illegible]	[illegible]	[illegible] 5/15	'30 [illegible]	[illegible] Yield to	[illegible] Stock
Co.	4¾	[illegible] '51, 6 '52	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	100¼	[illegible] maturity	[illegible] yield
	[illegible]	par '55	jA	'25	[illegible]	101¼	98¼	100¼	[illegible]	[illegible]	[illegible]

[illegible]

[illegible] theoretically and [illegible]

[illegible]

market quotation.

[illegible]

invested.

[illegible]

the capital intact.

[illegible]

A. Broker & Co.

Wall Street, N. Y.

[illegible]

Bought July 15, 1938

	Com- mission and shipping charge	Description	Interest and odd-lot fee
1	[illegible]	Western Union 5s	[illegible]

Mr. John Q. Buyers
Security, Vermont

[illegible]

Declared value up to:

[illegible]

[illegible] Broker's commission [illegible]

[illegible] coupon date arrived [illegible]

A(2). ***Buying a Bond at the Market—Theoretical Method.*** It is evident upon reflection that Mr. Buyers paid Mr. Sellers \$38.33 of accrued interest 42 days (180 − 138) before it was due and payable by the Western Union. Hence if he had wished to make the amount theoretically correct he would have transferred its dated value backward 42 days or $\frac{42}{180}$ of a coupon period to the delivery date by multiplying \$38.33 by the factor $v^{\frac{84}{360}}$ at the coupon rate.* This is found to give as the theoretical value of the accrued interest, $I_0 = 50 \times \frac{138}{180} \times 1/(1.025)^{\frac{84}{360}}$ or $38.33 \times .99425 = \$38.11$, which is the true delivery-dated value of the accrued interest due Mr. Sellers. This is the only actual theoretical difference between the practical and the theoretical methods of computation but it makes both the gross price and the total purchase cost different. The capital invested remains unchanged. Mr. Buyers' theoretical gross price would therefore be $2075 + 38.11 = \$2113.11$, and his theoretical total purchase price would be $2113.11 + 5.38 = \$2118.49$.

This entire transaction can now be put into general symbolic form as follows:

$$Q_0 = F \times \frac{q_0}{100} \qquad [\text{VI}\cdot 1]$$

$$G_0 = Q_0 + I_0 \qquad [\text{VI}\cdot 2]$$

$$T_0 = G_0 + B_0 = Q_0 + I_0 + B_0 \qquad [\text{VI}\cdot 3]$$

$$K_0 = Q_0 + B_0 = T_0 - I_0 \qquad [\text{VI}\cdot 4]$$

The above equations apply equally well to both the practical and the theoretical methods but the equation for the accrued interest for the practical method is:

$$I_0 = \frac{a}{t} \times D \qquad [\text{VI}\cdot 5]$$

in which a represents the number of days that have elapsed in the current coupon period before the delivery date, t the total time in days in the coupon period and D the value of one coupon.

---

*Note well that in order to use Table VII, the table of the fractional powers of v, the denominator of the fractional exponent of v must always be made 360 by multiplying both terms of the fraction by the same number.

For the theoretical method this equation for accrued interest becomes:

$$I_0 = \frac{a}{t} \times D \times v^{\frac{b}{t}} \tag{VI·6}$$

in which b represents the number of days from the evening before the delivery date to the next coupon date, found always by the equation $b = t - a$. Note that the interest rate used in the $v^{\frac{b}{t}}$ of this equation is here the coupon rate but in the computations of $B(2)$, for example, the yield rate will be used. One should therefore be on the alert to avoid using the wrong rate.

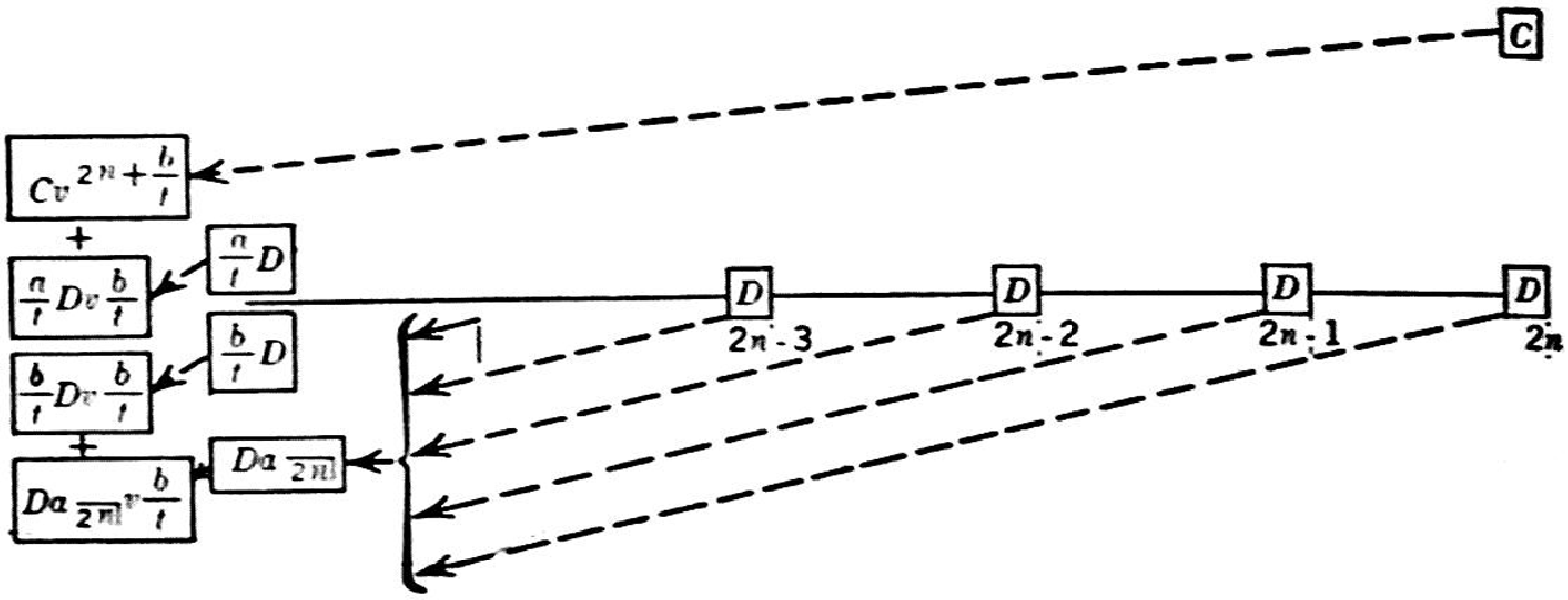

C
$Cv^{2n+\frac{b}{t}}$
+
$\frac{a}{t}D$
$\frac{a}{t}Dv^{\frac{b}{t}}$
$\frac{b}{t}D$
$\frac{b}{t}Dv^{\frac{b}{t}}$
+
$Da_{\overline{2n}|}$
$Da_{\overline{2n}|}v^{\frac{b}{t}}$
D
2n-3
D
2n-2
D
2n-1
D
2n

FIG. 11. A dated-value graph showing the purchase-dated value of the future benefits of a bond when the bond is to be held to maturity (theoretical method).

TABULAR VIEW OF SYMBOLS USED, TOGETHER WITH THE DEFINITION OF EACH

F = face value in dollars of the bond or bonds under consideration.

C = the maturity value or the redemption price of the bond or bonds.

D = the value in dollars and cents of one coupon or interest payment. It is computed at the coupon rate so that $D = F \times \frac{r}{2}$ in which

r_2 = the coupon rate, which is usually semiannually paid as indicated.

j_2 = the investment rate or "yield rate," i.e., the ratio that the remaining part of the coupons bears to the purchase price after certain deductions have been made in order to keep the capital intact.

I = that portion of a coupon that has accrued since the last coupon date; when subscripted with 0 the accrual period is the elapsed part of the current coupon period before purchase, when subscripted with a 1 the accrual period is the corresponding part of the coupon period within which resale takes place. This use of these two subscripts is general upon all symbols used in the same sense at both purchase and sale.

[illegible] BOND AT THE [illegible]

[illegible] the same subscript on both [illegible]

[illegible] a registration fee of 4 cents [illegible]

[illegible] capital investment at purchase [illegible]

K_1 = capital return at resale [illegible]

[illegible] date of delivery at purchase. [illegible]

[illegible]

EXERCISES

[illegible] On August 15, 1938 [illegible]

[illegible]

[illegible]

capital invested.

[illegible]

capital investment.

[illegible]

[illegible]

bond.

[illegible]

value P_p. He found that $P_p = 2000v^{44}_{.0225} + 50a_{\overline{44}|.0225}$ computed to this date at the yield rate, $j_2 = .04\frac{1}{2}$.

Making use of Table II and Table IV he found $P_p = 2000 \times .375677 + 50 \times 27.7477 = \$2138.74$. But P_p was seen to be dated March 1, 1938, instead of July 19, 1938, the delivery date. So Mr. Buyers computed the amount of P_p at simple interest for this interval of 138 days at the yield rate of $4\frac{1}{2}$ per cent. The sum, $P_p(1 + \frac{138}{180} \times .02\frac{1}{4})$, so found is the delivery-dated value of all future benefits of the bond including *all* of the current coupon. Hence it must be G_0 and therefore:

$$G_0 = 2138.74(1 + .0225 \times \tfrac{138}{180}) = \$2175.63 \text{ (see I}\cdot\text{1)}$$

But this is the gross purchase price because it contains the entire first coupon after purchase although $\frac{138}{180}$ of it has already accrued and belongs to the seller of the bond. So Mr. Buyers subtracted from G_0 $\frac{138}{180}$th of a coupon or $\frac{138}{180} \times 50 = \$38.33$, giving $Q_0 = G_0 - I_0 = 2175.63 - 38.33 = \$2137.30$.

He finally divided Q_0 by 20, giving the market quotation as $q_0 = 106\frac{7}{8}$, and he therefore concluded that he could afford to buy any bond having a 5 per cent coupon rate provided its market quotation was less than $106\frac{7}{8}$. The morning's quotation of that date (July 15, 1938) together with his knowledge about the safety and marketability of Western Union 5's were the determining facts from which Mr. Buyers decided to send the telegram to his broker to buy Western Union at the market.

The computations given above for this particular case may be generalized and expressed by the following formula. Let the student verify it.

$$G_0 = \left(Cv^{2n+1}_{\frac{j}{2}} + Da_{\overline{2n+1}|\frac{j}{2}}\right)\left(1 + \frac{a}{t} \times \frac{j}{2}\right) \qquad \text{[VI}\cdot\text{7]}$$

The value of Q_0 is then found by solving equation VI·2 for Q_0 and substituting into it the value of G_0 from VI·7 and that of I_0 from VI·5. The total purchase price and the capital invested may also be computed from the facts already obtained and the equations already given.

B(2). ***Buying a Bond on a Yield Basis—Theoretical Method.*** Had Mr. Buyers known the theoretical method of computing the market price to yield a predetermined interest rate on the money invested he would have found the present value of the same two benefits, the redemption price, C, and the annuity formed by the coupon payments but he would have found their purchase-delivery-dated values directly. He would also have found G_0 at one step by means of an equation that is only a little harder than VI·7. The derivation of this equation follows: using Mr. Buyers' $2000 bond the delivery date of the purchase of which was July 19, 1938, and the maturity date of which was March 1, 1960, we find that the delivery-dated value of the redemption price is:

$$2000v_{.0225}^{43+\frac{42}{180}} = 2000v^{43} \times v^{\frac{84}{360}}$$

The first coupon after delivery taken by itself has a delivery-dated value of

$$50v_{.0225}^{\frac{42}{180}} = 50v_{.0225}^{\frac{84}{360}}$$

and the remaining $2n$ coupons to final maturity constitute an annuity the delivery-dated value of which is:

$$50a_{\overline{43}|.0225} \times v_{.0225}^{\frac{84}{360}}$$

The sum of these three items gives G_0 directly as

$$\begin{aligned} G_0 &= (2000v_{.0225}^{43} + 50 + 50a_{\overline{43}|.0225})v^{\frac{84}{360}} = (2000 \times .384129 + \\ &\quad 50 + 50 \times 27.37203)\ .994822 \\ &= (768.258 + 50 + 1368.601)\ .994822 = \$2175.54 \end{aligned}$$

From this gross price, which is seen to be 9 cents less than the one which Mr. Buyers arrived at by the practical method, the market price or quoted price is obtained by the use of equations VI·6 and VI·2, giving:

$$\begin{aligned} Q_0 = G_0 - I_0 &= 2175.54 - \tfrac{138}{180} \times 50 \times .994822 \\ &= 2175.54 - 38.13 = \$2137.41 \end{aligned}$$

From this it is seen that the quoted price in this case is 11 cents more by the theoretical method than by the practical.

Again the solution of this particular example may be generalized and expressed symbolically to make other problems easier to solve:

$$G_0 = \left(Cv_{\frac{j}{2}}^{2n} + D + Da_{\overline{2n}|\frac{j}{2}}\right)v^{\frac{b}{t}} \qquad [VI\cdot 8]$$

BUYING BASED ON

rate used is $\frac{j}{2}$ not $\frac{r}{2}$.

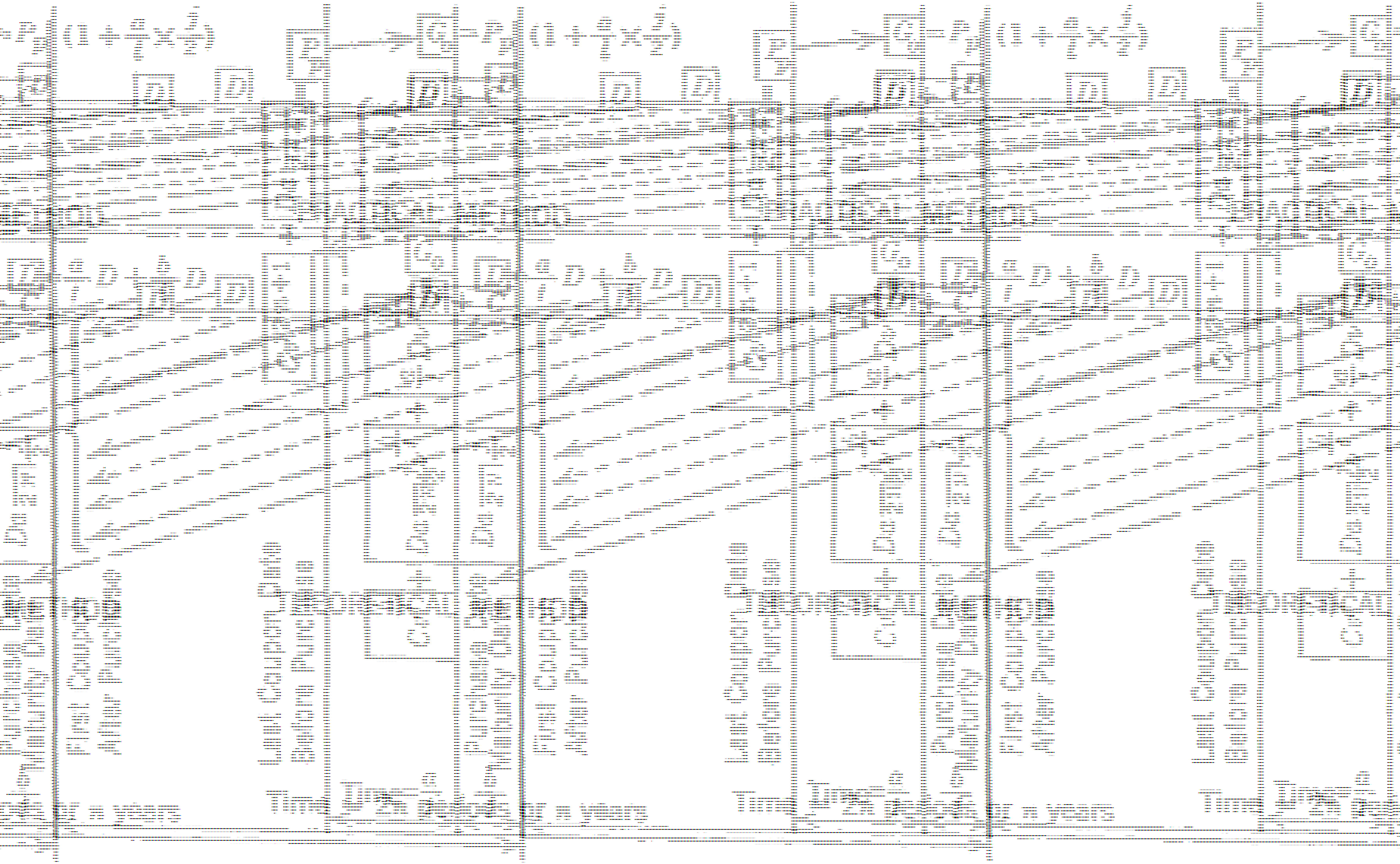


FIG. 12

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2166.68	2166.81	−.13	[illegible]	[illegible]	+.14	28.90	29.17	−.27
2170.70	2170.82	−.12	2137.62	2137.49	+.13	33.08	33.33	−.25
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible]

EXERCISES

[illegible]

[illegible] premium excess or premium [illegible]

[illegible]

[illegible] for the fact that [illegible]

[illegible]

Practical Methods [illegible]

[illegible]

[illegible] 12 to be, [illegible]

$$[illegible] \times 5.[illegible] \frac{1}{[illegible]_{.02}} = [illegible]$$

[illegible] intact, is:

$$[illegible] \, \frac{1}{[illegible]} \, P_1 = (Q_0 - C) \, [illegible]$$

[illegible] in its first five [illegible] The excess [illegible] Q_0 = [illegible]

[illegible] before, [illegible] would [illegible]

$$[illegible]$$

…o in Columns 2, 3, 4, 5, the Progress of Amortization of the Premium Paid for a $2000 Western Union …t at a Premium of $45.32 and Redeemed at Par; and in Columns 6, 7, 8, 9, the Progress of Accumulation …Discount of the Same Bond Bought at a Discount of $44.07 and Redeemed at Par

Book value of bond at beginning of each period	Periodic addition to depreciation fund	Total in the fund	Interest on capital	Book value of bond at beginning of each period	Periodic addition to depreciation fund	Total in the fund	Interest on capital
$2045.32			$2045.32	$1955.93			$1955.93
2036.6114	$8.7086	$ 8.7086	2045.32	1964.2308	$8.3008	$ 8.3008	1955.93
2027.7286	8.8828	17.5914	2045.32	1972.7806	8.5498	16.8506	1955.93
2018.6682	9.0604	26.6518	2045.32	1981.5869	8.8063	25.6569	1955.93
2009.4266	9.2416	35.8934	2045.32	1990.6574	9.0705	34.7274	1955.93
2000.0001	9.4265	45.3199	2045.32	2000.0000	9.3426	44.0700	1955.93

…o in Columns 2, 3, 4, 5, the Progress of Amortization of the Premium Paid for a $2000 Western Union …t at a Premium of $45.32 and Redeemed at Par; and in Columns 6, 7, 8, 9, the Progress of Accumulation …Discount of the Same Bond Bought at a Discount of $44.07 and Redeemed at Par

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2027.7286	8.8828	17.5914	2045.32	1972.7806	8.5498	16.8506	1955.93
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2009.4266	9.2416	35.8934	2045.32	1990.6574	9.0705	34.7274	1955.93
2000.0001	9.4265	45.3199	2045.32	2000.0000	9.3426	44.0700	1955.93

…o in Columns 2, 3, 4, 5, the Progress of Amortization of the Premium Paid for a $2000 Western Union …t at a Premium of $45.32 and Redeemed at Par; and in Columns 6, 7, 8, 9, the Progress of Accumulation …Discount of the Same Bond Bought at a Discount of $44.07 and Redeemed at Par

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2018.6682	9.0604	26.6518	2045.32	1981.5869	8.8063	25.6569	1955.93
2009.4266	9.2416	35.8934	2045.32	1990.6574	9.0705	34.7274	1955.93
2000.0001	9.4265	45.3199	2045.32	2000.0000	9.3426	44.0700	1955.93

[illegible]

EXERCISES

[illegible] (q.v.).

Ans. D_1 = $1.70

[illegible]

6. Using the corresponding results of the computations in Ex. 6, page 101, make out the first and last lines of a schedule by the theoretical method.

D(1a). ***Determinatiion of the Short-Term or Stock-Yield Rate on a Bond the Purchase of Which Has Been Made, or Is Contemplated at Market—Practical Method.*** Mr. Buyers, being something of a financial expert, before he sent off his buying order to Mr. Broker (see page 90), did a bit of research and figuring as to the probable financial advantages of his contemplated investment. First he followed the market price of Western Union back through the files of the past 3 years in *The Times* and found that it had remained steady at from 103 to 104 nearly all of this time. From this he reasoned that, should he wish to keep the bond only a short time, he could probably sell it at about the same price that he would have to pay for it if he bought now at the market. Assuming that he could sell at the same price he reasoned that his short-term yield would be the semiannually compounded rate found by dividing the semiannual coupon by the price paid. For this he used the quoted price, Q_0, giving him as the *stock yield* or *short-term yield* $50 \div 2075 = 2.41$ per cent per period or a nominal rate of $j = 4.82$ per cent, which is equivalent to an effective rate of $i = (1.024096)^2 - 1 = .0488$ per cent.

In general symbolic form, the *stock rate* is seen from this illustration to be

$$j_2 = \frac{2D}{Q_0} \quad \text{or} \quad i = \left(1 + \frac{D}{Q_0}\right)^2 - 1 \qquad [\text{VI}\cdot 10]$$

A moment of thought would make it evident that a more accurate and also a more conservative estimate would have been obtained had Mr. Buyers divided D by K_0 and used this result as the semiannually compounded rate. But this method is based on the variable market price in either case and such a refinement, while theoretically justified, is not practically defensible.

D(1b). ***The Long-Term Yield or Yield to Maturity When Purchase Price is Predetermined—Practical Method.*** But Mr. Buyers did not stop here. He planned to hold this bond to maturity, March 1, 1960, at which time it would be redeemed "at par" by Western Union. So he reasoned that if he paid \$2075 now and received \$2000 he would have an average of $(2075 + 2000) \div 2 =$ \$2037.50, capital invested during the period of ownership. But of the coupons received he must save out a total of $2075 - 2000$

[illegible]

[illegible] $\frac{48.2955}{2037.50}$ [illegible]

[illegible]

nal rate, j_2.

[illegible] general symbolic form:

[illegible]

[illegible]

[illegible] divided by E_0 [illegible]

$$\frac{C_j}{2E_0} \quad \frac{1}{(\text{zero})} \quad \frac{D}{E_0}$$

[illegible]

[illegible] symbolically this fact [illegible]

[illegible] interpolation equation [illegible]

$$\frac{D}{E_0} - \frac{C_j}{2E_0}$$

[illegible]

VI-12 becomes:

$$2000 \times \quad 150 \quad 75$$

[illegible]

[illegible] tion takes the form:

[illegible] .0306277 [illegible] 0 [illegible] .0377304 [illegible]

Hence

[illegible]

[illegible]

EXERCISES

[illegible] 5.413 per cent. [illegible]

[illegible] method. [illegible]

[illegible] per cent. [illegible]

[illegible] per cent. [illegible]

[illegible] Ex. 3, page [illegible]

[illegible] methods.

[illegible]

[illegible]

useful to warrant this

B. Trader & Co., Inc.,
Wall Street, N. Y.

Sold	Due

Registration [illegible] taxes	Commission Amount shipping	Interest Net amount fee
.80	.38	[illegible]

Mr. S. C. Sellers,
Ridley, Penn.

fore:

while the capital return

[VI·14]

and the accrued interest

$$I_1 = \frac{c}{t}D \quad [VI·15]$$

the accrued coupon

[VI·16]

EXERCISES

action in Ex. 2.

action in Ex. 2.

action in Ex. 2.

[illegible]

form and written:

[illegible]

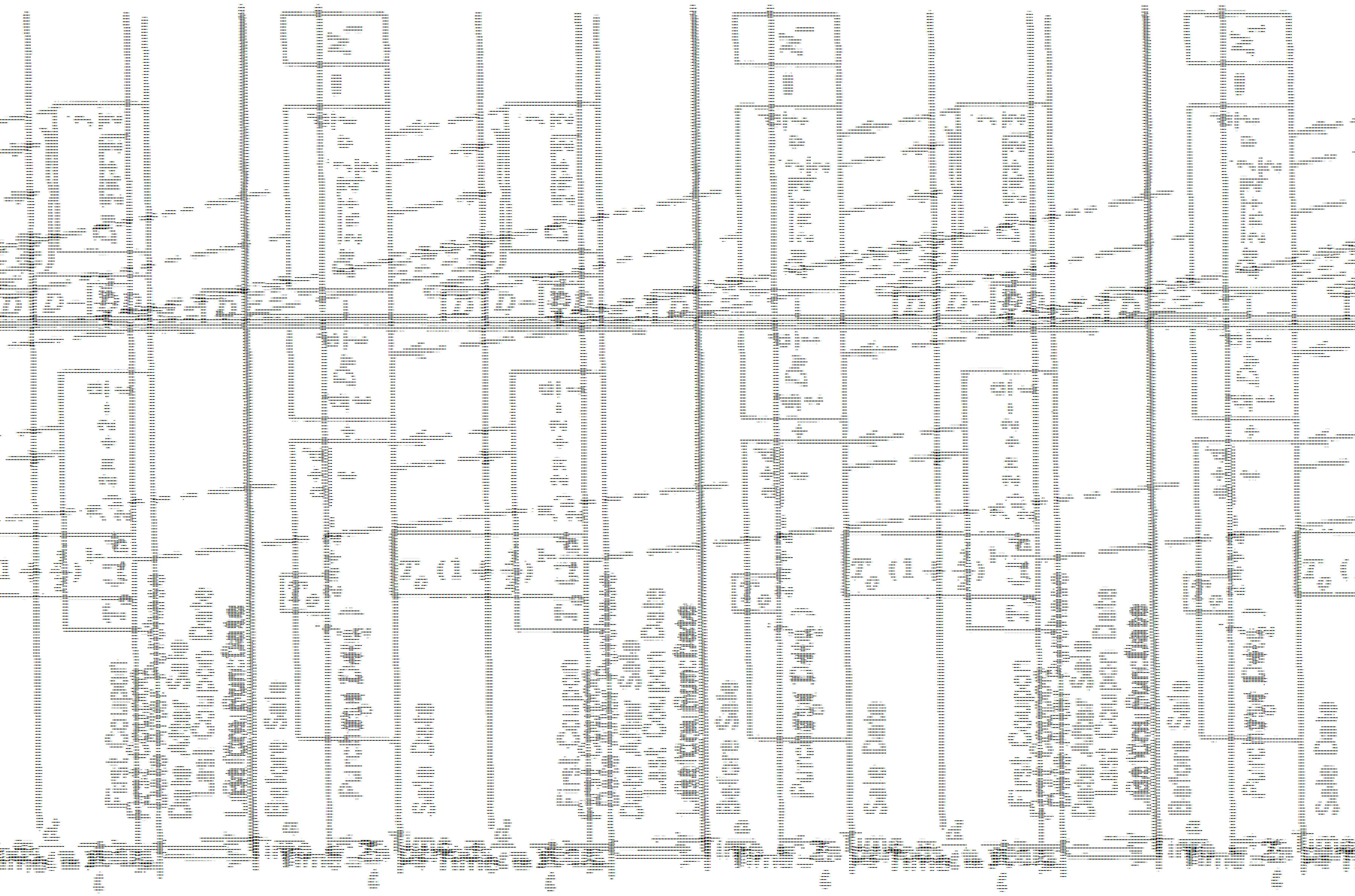

Fig. 13. [illegible]

EXERCISES

[illegible]

[illegible] 1923 [illegible]

[illegible]

accrued interest $3.88.

[illegible]

[illegible] = $506.86. (b) [illegible]

[illegible]

the yield [illegible] Remember that [illegible]

[illegible] 1938 [illegible]

[illegible] total purchase price [illegible]

[illegible]

[illegible]

computed.

[illegible]

as it was prospectively [illegible]

[illegible] 1938; [illegible]

[illegible]

$$\frac{j}{2} = \frac{48.24}{2084.325} = .0231$$

[illegible] form which follows:

$$\frac{j}{2} = \frac{D - \frac{T_0 - N_1}{2n + 1}}{\frac{1}{2}(T_0 + N_1)}$$

[illegible]

Multiplying the result [illegible]

[illegible]

function as:

[illegible]

[illegible] 138 [illegible] 180 [illegible]

[illegible] 36 [illegible] 180 [illegible]

[illegible] rates of [illegible] at all [illegible]

interest

[illegible] 50 [illegible]

[illegible] 50 [illegible]

[illegible] tion is:

[illegible]

Hence

[illegible]

[illegible] $i = 4.808$, effective.

EXERCISES

[illegible] for the period of ownership. [illegible]

[illegible] two transactions.

[illegible]

present value of the annuity consisting of the remaining $n - m$ installments, or:

$$V_m = Ra_{\overline{n-m}|i} = \frac{Fa_{\overline{n-m}|i}}{a_{\overline{n}|r}} \qquad \text{[VI·21]}$$

and the price $\frac{a}{t}$ of a period after the mth installment has been paid is found by the same reasoning as in the corresponding problem with a simple bond. It is:

$$Q_0 = V_m\,(1+i)^{\frac{a}{t}} - \frac{a}{t}\,Rv_i\,(1+i)^{\frac{a}{t}} = Rv^{\frac{b}{t}}\left[(1+i)a_{\overline{n-m}|i} - \frac{a}{t}\right] \qquad \text{[VI·22]}$$

Note that the interest rate in both VI·21 and VI·22 is the investment rate and not the coupon rate.

EXERCISES

1. A 10-year $10,000 annuity bond with interest at 6 per cent nominal, is to be paid in 20 semiannual installments. Find the purchase price at the end of the seventh year if bought to yield 5 per cent (semi). *Ans.* $3702.34

2. Find the purchase price of the above bond if bought 2 years earlier to give the same yield.

3. Determine the approximate yield rate if the bond had been bought as described but at a price of $3600. *Ans.* $j_{(2)} = 6.69$ per cent

4. Find the purchase price of a 10-year annuity bond for $4000 issued at a 4 per cent semiannual rate bought 4 years before maturity to yield 4 per cent (semi). *Ans.* $1792.01

5. Find the purchase price of the bond in Ex. 4 if bought to pay 5 per cent (semi).

6. Same as Ex. 5 except the yield to be 3 per cent (semi).

§ 31. **Serial Bonds.** Sometimes a given issue of bonds contains a provision for the redemption of stated portions of the issue at various stated dates instead of having but one redemption date for the whole issue. Such an issue is called an issue of *serial bonds.* When such an issue is sold to a single banking house it becomes necessary to compute a purchase price to give a predetermined yield.

To do this the excess of each part of the issue is computed as in (b), page 108, and the total excess for the entire issue is found by simply summing the separate excesses. The purchase price will be the total redemption price of the issue plus the total excess.

[illegible] (b), page 108, to be:

$$E = \left(D - C\frac{j}{2}\right) [illegible]$$

[illegible]

EXERCISES

[illegible]

MISCELLANEOUS EXERCISES

[illegible]

sold August 6, 1940, in order to yield an income rate of 6 per cent (semi)? Broker's charges at purchase were \$2.74 and at sale, \$3.06.

Ans. $N_1 = \$968.76$

4. The purchaser of the above French bonds on July 8, 1938, intends to hold them until maturity in 1941, when he will redeem them. What income rate will he have earned on his investment? *Ans.* $j = 6.14$ per cent

5. A Boston Elevated Railway Co. $4\frac{1}{2}$ per cent '41 mN NC bond for \$5000 was bought June 16, 1930, at 93. This bond was originally sold in 1911 at par. What investment rate did an original owner of this bond who sold at the market on June 16, 1930, earn on his investment? Take B_1 as \$10.74.

6. On May 15, 1930, the South Puerto Rico Sugar bond of Ex. 5 on page 96 was sold at the market (105); accrued interest less commission and shipping charges, \$2.60; transfer tax of 40 cents and registration tax of 4 cents. Make out the bill for the sale of this bond as sent by the broker to the seller. What were the gross selling price, the net selling price and the return of capital?

7. Suppose that the man who bought the Denmark $5\frac{1}{2}$ per cent '55 on March 18, 1938 (see Ex. 1, page 101), were to sell it on December 4, 1944, so as to make a yield rate of 6 per cent for the time that he had owned it just as he had planned. Find the net sale price, the quoted price and the capital return.

8. A \$5000 International Paper Co. 6 per cent '55 Ms bond was bought July 6, 1928, at a total purchase cost of \$5370.94 and sold on September 10, 1937, at a quoted price of $95\frac{1}{4}$. Compute the following items: (a) the accrued interest at purchase, (b) the accrued interest at sale, (c) the approximate brokerage charges at sale, (d) the approximate brokerage charges at purchase. The bond was bought and sold on the New York exchange (in third zone) through a member of the exchange as broker. Value declared and privately insured. *Ans.* $I_0 = \$107.50$, $I_1 = \$10.83$, $B_0 = \$10.68$, $B_1 = \$12.72$

9. A \$1000 N. Y. C. R. R. 4 Fa '98 was bought September 10, 1937, at a quoted price of 95 with $B_0 = \$2.65$ and sold February 1, 1938, at a quoted price of 84 with $B_1 = \$3.09$. Compute the gross purchase price, the total purchase cost, the capital investment, the net sale price and the return of capital. Was this a good investment? What income rate do you consider this investment as yielding? *Ans.* (a) $G_0 = \$954.78$, (b) $T_0 = \$957.43$, $K_0 = \$952.65$, $N_1 = \$837.13$, $K_1 = \$836.91$

CHAPTER VII

[illegible]

[illegible]

build a home borrow to build a home

[illegible]

unmatured shares.

[illegible]

shares.

[illegible]

any time.

[illegible]

building.

[illegible]

investor.

$1.050625 - 1 = .050625.$

[illegible] the [illegible]
[illegible]
month schedule [illegible]

[illegible]	Interest on current value of share, for current month	Book value of share at beginning of month
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]

199.75 × 100 [illegible]

[illegible] 250 [illegible] 04 30 [illegible]

[illegible] 12 [illegible]

[illegible]

[illegible] days. 1 month, 10 days.

[illegible]

schedule.

[illegible] paid monthly [illegible]

[illegible]

is paid

[illegible] this formula in general [illegible] payment periods [illegible] it is:

$$\frac{F}{6Ms_{\overline{n}|\frac{i}{2}}^{(6)} \times \left(1+\frac{i}{2}\right)}$$

Table XIII, last column [illegible]

[illegible]

[illegible] rough comparison.

[illegible]

[illegible] is the approximation method [illegible]

[illegible]

Beginning of each month starting at beginning of second half of 13th year	Book value of the share before adding dues current month	Amount of the dues paid at beginning of month	Book value of share after addition of month's dues	Interest on current book value, for the month
$12\frac{1}{2}$ yrs., 1st of 7th month	$194.4177	$1.0000	$195.4177	.6514
at the 1st of 8th month	196.0691	1.0000	197.0691	.6569
at the 1st of 9th month	197.7260	1.0000	198.7260	.6624
at the 1st of 10th month	199.3884	.6116		

Hence in this case the time for maturing the share is 12 years, 9 months and 61 cents of the dues payable at the beginning of the tenth month. The time to mature a share is either a number of years, months and days or a number of years, months and some portion of the next month's dues.

EXERCISES

1. When the Middlebury Building and Loan Association began by maturing shares to $200 by $1 per month dues and a semiannual dividend at 6 per cent, how long were the shares in maturing and what effective rate did the investor's money earn? *Ans.* 11 years, 6 months, 27 days; $i = 6.09$ per cent

2. The rate in the above association was reduced to $4\frac{1}{2}$ per cent after a few years. How long did it then take shares to mature and what was the investor's rate under those conditions? *Ans.* 12 years, 5 months, 68 cents of next dues

3. Now it has been still further reduced to 4 per cent. What is the effective investor's rate now and how long does it take to mature a share?

4. Connecticut State laws allow associations to mature shares to four different values: $100, $200, $300 and $400. If the dividend rate to investors is 5 per cent and the dues $1 per month, what is the maturing time in each of these four kinds of shares? *Ans.* 12 years, 1 month, 10 days

5. An association maturing $200 shares by $1 per month dues changes its semiannually compounded interest rate from $4\frac{1}{2}$ to $3\frac{1}{2}$ per cent. How much did this lengthen the maturing time? *Ans.* 8 months and 32 cents dues

6. An association officer wished to know what effect it would have upon the maturing time if the dues were changed through a series of values, all of which are in use somewhere: (a) 50 cents, (b) $1, (c) $1.50 and (d) $2. If $200 shares are matured at a semiannual rate of 5 per cent what are the four answers to this official's inquiry? Use any result already available.

The Borrower's Real Interest Rate. The borrower in this type of association pays dues on shares equal in face value to the amount of his loan and in addition he pays interest monthly in advance on the face of his loan until the shares for which he is paying mature. These matured shares are regarded as the payment in full of his loan. From his point of view, therefore, his monthly dues plus interest on his loan constitute a monthly paid annuity due, the present value of which equals the face of the loan, provided the interest rate used to bring it back to the present is the actual effective rate that he is paying for his loan but *not* the paid-in-advance rate described above. The term of this annuity is the maturity time of all shares as already found for the investor. It is this actual interest rate that interests the borrower. This is what will now be found. On page 49 it was shown that an interest rate compounded with the same frequency as the payments of the installments of rent of the annuity is much simpler to find. We shall therefore find $j_{(12)}$ and from it find the equivalent effective rate.

Let $r_{(12)}$ = the quoted paid-in-advance rate charged the borrower

and $I = F\frac{r}{12}$ = the month's installment of interest paid by the borrower

$j_{(12)}$ = actual nominal monthly compounded rate paid by the borrower

c = the number of cents of the last dues when c is the sum needed to mature the share

or d = the number of days in the final time (partial month) needed to mature the share when a partial dividend payment is the thing needed to mature the share

v_1 = the discount factor at a rate 1 per cent greater than $r_{(12)}$. (This rate is chosen because it removes one unknown factor from VII·3 and VII·4 and has been found by many trials to give extremely accurate results.)

n = years and fraction of a year which expresses the number of complete months in the maturing time of the share

All other symbols have the same meaning as previously defined.

Two cases arise:

[illegible]

this last payment:

$$[illegible] 12(M + I)a^{(12)}_{\overline{n}|(12)}$$

[illegible]

$$[illegible] \left(M + \frac{d}{30} I\right) [illegible] a^{(12)}_{\overline{n}|(12)}$$

[illegible] month's interest [illegible]

[illegible] equation in useful form [illegible]

$$[illegible] \left(M + \frac{d}{30} I\right) [illegible]$$

[illegible] EFFECTIVE RATE [illegible]

[illegible] (Case I), page 125 [illegible] the nominal [illegible] borrower [illegible] rate did the [illegible] actually pay?

[illegible] solution [illegible] Problem [illegible] $200 [illegible] years [illegible]

[illegible] into formula VII-3, giving

$$\left(1 + \frac{.05 + .01}{12}\right)$$

[illegible] .8333333 [illegible] 1 + .8333333 [illegible]

[illegible] The interpolation [illegible] referred to on page [illegible]

[illegible] per cent [illegible] 106.2894237

$$a_{\frac{j}{12}} = 107.9357856$$

$$a_{\frac{5}{12}} = 112.4355942$$

Hence the interpolation equation

$$\frac{\frac{j}{12} - \frac{5}{12}}{[illegible]} = \frac{[illegible] - 107.9357856}{[illegible] - 106.2894237}$$

[illegible] whence [illegible] .4776777 [illegible] per cent [illegible]

.004776777.

The effective rate is now [illegible] Hence $i =$ .05 [illegible]

PROBLEM [illegible] (CASE II). In Problem 12 consider [illegible] the borrower [illegible] rate did he actually pay?

SOLUTION. The data [illegible] $200 [illegible]

[illegible] steps to the student.

[illegible] 12

EXERCISES

[illegible] puted in a previous problem [illegible]

[illegible]

Solution. The bookkeeper would make out a schedule like the portion of one shown below. No explanation other than that furnished by the headings will be needed.

Beginning of month, after the payment	Book value of the debt at beginning of each month	Total monthly payment	Interest on current book value of debt at $\frac{1}{2}\%$ per month	Part of payment left for amortization
1	100.0000	1.0000	.5000	.5000
2	99.5000	1.0000	.4975	.5025
3	98.9975	1.0000	.4950	.5050
4				
137	2.9653	1.0000	.0148	.9852
138	1.9801	1.0000	.0099	.9901
139	.9900	1.0000	.0049	.9951*

* By this table we see that the 139th monthly payment of $1 pays the interest, $.0049, and leaves .9951 with which to amortize the remaining book value of .9900, or .0051 more than needed. It therefore takes 139 months or 11 years, 7 months, to pay the loan.

The above schedule is 139 lines long and requires great patience for its complete computation but the amortization formula gives a direct method for finding the term of the loan when it has been solved for $a_{\overline{12n}|\frac{j}{12}}$. The derivation of the formula is as follows: $F = 12Ma^{(12)}_{\overline{n}|j_{(12)}} = Ma_{\overline{12n}|\frac{j}{12}}$; hence the appropriate form for the formula is

$$a_{\overline{12n}|\frac{j}{12}} = \frac{F}{M} \qquad \text{[VII}\cdot\text{5]}$$

Applying this formula to the problem just solved we may get a shorter second solution:

$$a_{\overline{12n}|\frac{j}{12}} = a_{\overline{12\cdot n}|.005} = \$100$$

Looking this value up in Table IV under the column headed $\frac{1}{2}$ per cent we find that for $n = 139$, $a_{\overline{12n}|.005} = 100.0121$, hence the time is 139 months or 11 years, 7 months, accurate to much [illegible] LOAN [illegible]

[illegible]

page 47.

[illegible]

[illegible] equivalent to $i =$ [illegible]

[illegible] equivalent to $i =$ [illegible]

[illegible] equivalent to $i =$ [illegible]

[illegible] equivalent to $i =$ [illegible]

[illegible] equivalent to $i =$ [illegible]

[illegible]

$$= 12n = \frac{P}{M} \qquad [\text{VII}\cdot 6]$$

$$B_k = P - kM \qquad [\text{VII}\cdot 7]$$

$$\frac{j}{12}\,[P - (k \quad [illegible]$$

[illegible]

EXERCISES

[illegible]

4. You have doubtless been impressed already by the fact that the direct-reduction type of mortgage is very much more understandable than the other, which we might call the sinking-fund type, since interest on the face is paid until the date of settlement. Could you not simplify the form of explanation of this latter type so that it would be much more easily understood? Try it and submit your verbal statement.

5. A modification of the direct-reduction mortgage was used at one time by Sears Roebuck and Company in financing their "ready-built" houses. By this plan the monthly payment on a loan of $4000, say, was to be $20 plus the interest at 6 per cent on the outstanding book value at the end of each month, the $20 being applied directly to reduce the book value of the loan. What is the first month's payment? How much is it reduced each month? What is the term of the loan? What is the final payment? What effective rate is the borrower paying? *Ans.* (a) $40, (b) 10 cents, (c) 200 months, (d) $20.10, (e) .06167

MISCELLANEOUS EXERCISES

1. A certain building and loan association was forced by the depression to reduce its dividend rate to investors to 3 per cent paid semiannually. Its shares were matured to $200 by monthly-in-advance payments of $1 and these dividends. In what length of time and by what final payment, if any, were these shares matured? *Ans.* 13 years, 6 months, and 36 cents

2. The building and loan association in Ex. 1 operated upon a nominal spread of 1 per cent. This actually means that it charges borrowers nominally 4 per cent interest which is paid monthly in advance along with the monthly dues of $1 per share. Find the actual effective interest rate paid by the borrower in this association. *Ans.* 4.758 per cent

3. What is the actual effective "spread" between the investor's and the borrower's rate in the above association? *Ans.* 1.894 per cent

4. Were nominal interest rates to investor and borrower to remain fixed at any predetermined values subject only to the condition that the borrower rate is the larger, what effect upon the borrower's real effective rate would an increase in the monthly dues make? Carry this process to its logical conclusion and then explain why such associations are ever a help to the borrower.

5. In explaining to the investor how his dividend is computed the loan officer states that the first month's dues, $1, is in use by the association for 6 months, the second for 5 months and so on, until the last which is in use only 1 month. Hence the association has the use of the equivalent of $1 for $6 + 5 + 4 + 3 + 2 + 1 = 21$ months or $\frac{21}{6} = \$3.50$ for 6 months in addition to the use of all previously paid dues for the full 6 months. The sum of these two items plus the accumulations from compoundings is the principal on which they compute dividends. All this is true but complicated. It would, however, serve well another purpose. During the time that one series of shares takes to mature, a surplus of some size accumulates in the hands of the association. This surplus really belongs to the share owners, and when a series matures the just proportion of this undistributed profit should be credited to the maturing share owners. Assuming all the data and results of the first three problems of this set, see if you can work out, on the basis of the relative

use-value of the several series in the hands of the association, a just amount to be allotted to each maturing share. Assume 100 shares in each series and that shares mature in 12 years. *Ans.* .07376 per cent of surplus par shares

6. If a building and loan association requiring $1 per month dues payable in advance matures shares to $200 each in exactly 12 years, what is the semi-annually compounded rate of interest paid by the association to its investors?

7. If the association described in Ex. 6 has an average of 1200 shares per series including both investors' and borrowers' shares, all earning income for the association on a 1 per cent interest-rate spread, how much undivided "profit" per year is the association making? *Ans.* $28,899.61

8. A building and loan association had been operating on the sinking-fund or pseudo-cooperative plan paying investors 5 per cent in semiannual installments and operating on a nominal 1 per cent spread. They finally decided to take out a federal charter and write direct-reduction mortgages at 6 per cent nominal, paid at the ends of months. What effect did this have on the effective rate of income earned if investor's rate remains unchanged?

Ans. Increased by .761 per cent

CHAPTER VIII

CHOICE—THE FOUNDATIONS OF PROBABILITY

Life annuities and life insurance both depend upon probability; in the one case, that of being alive and in the other of being dead. It is therefore necessary to study the fundamentals of probability at this point.

Probability in turn depends upon the number of ways in which a definite number of objects can be chosen from a group of objects. Hence it is necessary to study first of all the question of how objects may be chosen.

§ 34. **Choice or Permutations and Combinations.** When a question is asked which involves the number of ways in which a group of objects can be chosen or arranged, two possibilities arise: (1) either every order in which any particular group of objects may be chosen or arranged shall count as a different way of choosing or (2) only one order or choice of any particular group shall be counted and all other orders, as long as they contain exactly the same objects, shall be considered to be duplicates of this order and therefore not counted. In the first of these cases each of the ways of choosing is called a *permutation* and in the second it is called a *combination.*

We may formally define a *permutation* as a group of objects chosen or arranged in a definite order. It is evident that any change in any one or more of the objects or any change in the order of the objects would in either case result in a new permutation and so count as a different way of choosing.

We may define a *combination* as a definite group of objects regardless of the order in which they were chosen or arranged. It is therefore evident that no change in the order that left all the objects of the group unchanged could be counted as a different combination, but the replacement of even one object of the group by a new one would produce a different way of choosing whenever the choosing was to be done by combinations.

Since it is always necessary to choose objects in some one order

[illegible]

[illegible] first.

[illegible] the n objects we see [illegible]

[illegible] (read "factorial n")

[illegible] n factors. Instead of n factors [illegible]

[illegible] written more briefly as

[illegible]

remaining four in second place, any one of the remaining three in third place, any one of the remaining two in fourth place, and the last one in the fifth place. Hence all five can be placed in:

$${}_5P_5 = 5 \times 4 \times 3 \times 2 \times 1 = \underline{|5} = 120 \text{ orders or permutations.}$$

Suppose they had been taken three at a time; in how many orders could they be placed? The same reasoning would be applied but only three factors would be used. Hence:

$${}_5P_3 = 5 \times 4 \times 3 = 60 \text{ ways, orders or permutations.}$$

The method used in these problems can be put into a general statement: *Fundamental Principle. If a thing can be done in* **m** *ways and if after it has been done in one of these ways another can be done in* **n** *ways, then the two can be done together in that order in* **mn** *ways.* This principle can be extended to finding the number of ways of doing three or more things together.

In many cases the nature of the problem is such that the "ways of choosing" must be by combinations. For example, if you are asked how many different committees of three can be chosen from four persons, a moment's thought will convince you that the order in which the committee is chosen or arranged has no effect upon the number of committees. It will be the same committee as long as it has the same individuals on it regardless of the order in which they are chosen. Each different group of three must be counted only once. But if the number of permutations of four things taken three at a time is computed each different group of three is counted as ${}_3P_3 = 3 \times 2 \times 1 = 6$, while the total number of permutations of four things taken three at a time is ${}_4P_3 = 4 \times 3 \times 2 = 24$. But only one out of every six of these is a different combination since each combination has six different permutations. Hence the number of combinations of four things taken three at a time is:

$${}_4C_3 = \frac{{}_4P_3}{{}_3P_3} = \frac{24}{6} = 4$$

The first part of this formula serves as the general formula for computing the number of combinations of n objects taken r at a time.

$${}_nC_r = \frac{{}_nP_r}{{}_rP_r} = \frac{\underline{|n}}{\underline{|n - r} \times \underline{|r}} \qquad \text{[VIII·4]}$$

group of r objects.

EXERCISES

many weeks will it take?

Ans. 10 weeks

every other one of them?

determine a circle?

different groups, and the general principle enables one to compute the total number of ways of choosing objects from two or more independent groups.

Since the choosing from one group may be by combination and that from the other by permutations, or both by combinations or both by permutations, and in addition to this there may or may not be permutations between the two groups, several cases may arise.

Case I. When the choosing from both of the two groups is by unrestricted combinations.

Illustration. The first nine letters of the alphabet contain three vowels and six consonants. How many *different* groups of letters containing two vowels and three consonants can be chosen from the first nine letters of the alphabet? The number of choices of two vowels out of a group of three is ${}_3C_2 = \dfrac{3 \times 2}{2 \times 1} = 3$. The number of choices of three consonants out of a group of six is ${}_6C_3 = \dfrac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$. Hence the total number of different groups that can be made up from these nine letters to contain three consonants and two vowels is ${}_3C_2 \times {}_6C_3 = 60$.

Case II. When the choosing from both groups is by unrestricted permutations.

Illustration. Suppose that after the above letters had been chosen they had been arranged in all possible orders, how many possible orders were there? This is a problem in unrestricted permutations and the number would be found by permuting one group and multiplying by the number of groups as found in the first case. Total number is:

$$
\begin{aligned}
{}_6C_3 \times {}_3C_2 \times {}_5P_5 &= \frac{{}_6P_3 \times {}_3P_2 \times {}_5P_5}{{}_3P_3 \times {}_2P_2} \\
&= \frac{(6 \times 5 \times 4)(3 \times 2)(5 \times 4 \times 3 \times 2 \times 1)}{(3 \times 2 \times 1)(2 \times 1)} \\
&= 20 \times 3 \times 120 = 7200
\end{aligned}
$$

Case III. When the choosing of each group is by unrestricted permutations but the members of one group are not to be permuted with those of the other.

[illegible]

[illegible] $_6P_3$ [illegible]

[illegible]

sonants by permutations and after being so chosen and put together no further arrangement of them is made. Hence the answer is ${}_3C_2 \times {}_6P_3 = 3 \times 120 = 360$.

Case VI. When the members of one group, say the first group, are not permuted among themselves but those of the second group are permuted both among themselves and with those of the first group in so far as such permutation preserves the order of the members of the first group.

Illustration. Given the same letters from which to choose and the same sized groups to be chosen, how many words can be made of all collections of the five letters so long as the vowels occur in their natural alphabetical order? It is evident that the vowels must first be chosen by combinations and the consonants by permutations and then the product so formed will need to be multiplied by the number of ways we can combine five places two at a time, or five places three at a time. Hence the total number of words is ${}_3C_2 \times {}_6P_3 \times {}_5C_2$ or ${}_3C_2 \times {}_6P_3 \times {}_5C_3 = 3 \times 120 \times 10 = 3600$.

For two groups this shows the various possibilities and the same reasoning applies to three or more groups. These illustrations show to a certain extent the necessity for careful analysis of the particular problem proposed for solution.

The most important application of "choice" is in the subject of probability, which is taken up in Chapter IX.

EXERCISES

1. Find the numerical values of the following: ${}_5C_3 \times {}_4C_2$, ${}_3P_2 \times {}_4C_3 \times {}_3C_3$, ${}_4P_3 \times {}_5P_1 \times {}_3P_3$; ${}_5P_4 \times {}_4P_3 \times {}_3C_2 \times {}_4P_0$ and ${}_6P_0 \times {}_5C_0 \times {}_4P_2$ using all available results from Ex. 1 on page 139.

2. Express the following in the form of a verbal problem, using any two groups of objects about which you can devise problems (try to be as original and interesting as possible): (a) ${}_5P_2 \times {}_4P_3$, (b) ${}_5C_2 \times {}_4P_3 \times {}_5C_3$, (c) ${}_5C_2 \times {}_4P_3$, (d) ${}_5C_2 \times {}_4C_3 \times {}_7P_7$.

3. At a stag banquet of 16 people the toastmaster sits at the head of the table with the guest of honor at his right. Other guests are seated by a committee of arrangements. In how many ways could this task be accomplished?

4. A baseball nine enters a bus in which there are twelve vacant seats. In how many ways can they be seated?

5. From a group of ten juniors and eight seniors a committee of six consisting of three from each class is to be drawn. In how many ways can such a committee be chosen? How many different committees can be chosen?

6. From a group of four men and five women a reception committee of three

[illegible] chairman [illegible] is to be [illegible]

[illegible]

[illegible] return wire?

[illegible]

[illegible] must remain in order [illegible]

[illegible] ones can be [illegible]

[illegible]

MISCELLANEOUS EXERCISES

[illegible] arranged? *Ans.* 5040.

[illegible]

[illegible] prizes be awarded? *Ans.* 90.

[illegible]

[illegible] may they [illegible]

[illegible]

[illegible]

Ans. 2772

[illegible] on pages 140, 141 and 142 [illegible]

[illegible]

Ans. 69

[illegible] letters? [illegible] person, post [illegible]

[illegible] of the total number of [illegible]

[illegible] you discover, whether [illegible]

[illegible] How many of these [illegible] order? [illegible]

CHAPTER IX

PROBABILITY—ONE OF THE FOUNDATIONS OF LIFE ANNUITIES AND LIFE INSURANCE

§ 35. **Fundamentals.** Probability used to be called "chance" and it still suffers scars upon its reputation from the evil association of its name with "games of chance." In point of fact probability is a science, a thorough knowledge of which would cure even the most confirmed gambler of his delusion that he can "break the bank at Monte Carlo."

The probability that an event can happen or succeed is defined as the quotient of the number of ways in which it can happen or succeed divided by the sum of the total number of ways in which it can happen or succeed plus the total number of ways in which it can fail to happen or succeed. Symbolically,

$$p = \frac{s}{s+u} \qquad [\text{IX}\cdot 1]$$

an equation in which p represents the probability fraction, s the number of successful choices and u the number of unsuccessful choices. It becomes evident at once that choice finds here an application directly in finding both numerator and denominator of the probability fraction. The statement of each problem will reveal, if carefully analyzed, which of the two methods of choice, permutations or combinations, is applicable in that problem. It is evident that when and only when all of the ways of choice become successful does this fraction of IX·1 become integral, and then shows certainty, the probability of all possibilities.

If, instead of using the theoretical approach indicated in the previous paragraph, we cause the happening of the event in question several times in succession and record the number of successes and the total numbers of tries and, if we use the above equation, we call the fraction obtained the *relative frequency* of the success of the event. If the event is of such a kind that we can reason out the values of s and u we shall find that the value of p, the probability, is not in general the same as the value of the relative frequency found by actual experiment. If, however, we conduct a sufficient number of experiments and examine the value of the relative frequency found from an ever-increasing number of experiments we shall discover that it steadily draws nearer to the value of p found by theory. By making the number of experiments sufficiently large we can make the experimental value of the relative frequency approach the theoretical probability as nearly as we may wish.

Since relative frequency becomes probability for a large number of repetitions in such cases as can be tested by theory, we have assumed that it will do the same in a great number of types of problems in which there can be no theoretical verification. Life insurance is based upon the probability at each age that a person will live another year or a certain number of years and this may be determined by the relative frequency of death in each age group, as the size of the group is made as large as possible.

The expected number of occurrences of an event when n trials are made is the product of the probability of its happening multiplied by n, the number of trials:

$$\theta = np$$

In the same way the monetary value of a prize-winning is the product of the cash value of the prize multiplied by the probability of winning it:

$$M = cp$$

EXERCISES

1. What is the probability that a date chosen at random is a Sunday, that it is a day in March, that it falls on the tenth of a month, that it is Christmas day?

2. An autoist comes to an unmarked "four corners." He wishes to go to a certain town. What is the probability that he will get there at the first try?

3. A date is chosen at random repeatedly. When this has been done 546 times how many Sundays can be expected to be among the number chosen?

4. An article worth $10 is being raffled off. One hundred tickets selling for 50 cents each are sold. What is the monetary value of the expectation of winning to a person holding one ticket? *Ans.* 10 cts.

5. If it were equally probable that tomorrow be fair as that it be rainy, how many rainy days would there be per year?

6. Out of 1000 cars passing a certain traffic counter 586 were Fords. Can you predict how many of the next 40 cars will be Fords? *Ans.* 23 or 24

7. What is the probability that a certain player, Jones, will head the batting list of the baseball team to which he belongs?

8. Ten spellers, all winners of all previous contests in which they had participated, were to enter as competitors in a final contest. What chance has a predetermined one of them to win?

9. A certain club has 40 members of whom 28 are whist players. The three earliest arrivals one evening decided on a game of whist and awaited the arrival of a fourth. What is the probability that he was a whist player? *Ans.* $\frac{13}{20}$

10. One complete alphabet, each letter on a little square card, was put in a box and letters drawn one at a time. The first letter drawn was a *b*. What was the probability of this event? The second was a *g*. What was the probability of this event if the first was retained? Only by drawing a vowel for the third letter could a three-letter word be made. What was the probability of drawing a vowel? Of making a three-letter word?

§ 36. **Compound Probability.** But the player who tries to break the bank at Monte Carlo never expects to do it at a single play and so, unfortunately, does not content himself with one attempt. As his number of plays increases his probability problem grows more and more complex.

When the probability to be computed is that of the occurrence or success of a single simple event there is no chance of going wrong, since both numerator and denominator of the probability fraction will be the simple permutations or combinations that comprise the successful and the total choices or ways in which the event can occur. When the event under consideration is comprised of two or more single events which might possibly occur together or in succession, three types of situations arise which require separate consideration.

Independent Events. Two or more events are independent if the occurrence of any one of them at any trial does not affect the probability of occurrence of the others.

For example, if the probability that a person 49 years old will live a year is .7 and you have three friends all of age 49, the probability of living for any one of them is evidently independent of that of the others. We may show that the probability that all three will live a year is the product of $.7 \times .7 \times .7 = .343$.

General Proof. Let m_1, n_1, and r_1 be the respective number of ways in which three independent events can succeed and let

[illegible]

[illegible]

$$\frac{m_1 \times n_1 \times r_1}{(m_1 + m_2)(n_1 + n_2)(r_1 + r_2)}$$

$$[illegible] \frac{m_1}{m_1 + m_2} \times \frac{n_1}{n_1 + n_2} \times [illegible]$$

[illegible]

[illegible] dependent; they are independent [illegible] *Dependent Events.* [illegible] affects the probability [illegible] one of them [illegible] the occurrence of the other. [illegible] letters of the alphabet [illegible] For example, if you [illegible] probability of your selecting [illegible] from which [illegible] selected letters [illegible] three vowels [illegible] between selections? [illegible] different ways. [illegible] For [illegible] two successive [illegible] second [illegible]

$$\frac{3}{9}\times\frac{2}{8}\times\frac{1}{7}=\frac{1}{84}.$$

[illegible]

$\frac{1}{84}$ as before. $6\times\frac{1}{504}=\frac{1}{84}$

[illegible]

[illegible] used in [illegible]
[illegible] question involved in [illegible]
[illegible] example [illegible]
[illegible] only [illegible] which there is only [illegible]
[illegible] probabilities [illegible]
[illegible] probability [illegible]
[illegible] mutually [illegible]
[illegible] exclusive [illegible] particular [illegible]
[illegible] other [illegible]
[illegible] Christmas falls on a [illegible]
[illegible] Monday [illegible]
[illegible] Tuesday or a Saturday [illegible]
[illegible] sure [illegible]
[illegible] probability [illegible]
[illegible] which we can [illegible]
[illegible] fundamental theorem [illegible]
[illegible] mutually exclusive [illegible]
[illegible] probability of failure [illegible]
[illegible] success and [illegible]
[illegible] the [illegible]
[illegible] Hence [illegible]
[illegible] the certainty [illegible]
[illegible] one or [illegible]
[illegible] sum of all the possibilities [illegible]
[illegible] in general [illegible]

[illegible] to one and to certainty [illegible]
[illegible] illustrated [illegible]
[illegible] proof as follows [illegible] general proof [illegible]
[illegible] If the [illegible]

[illegible] under condition of which [illegible]

[illegible] unity and we can write

$$[illegible]$$

[illegible] question in the problem

$$[illegible]$$

$$[illegible]$$

[illegible] in the expanded form of [illegible]

[illegible]

[illegible]

a Sunday Saturday or a Sunday Saturday or a Sunday Saturday or

[illegible]

$+ \frac{1}{343} + \frac{3}{343} + \frac{3}{343} + \frac{1}{343} + \frac{3}{343} + \frac{3}{343} + \frac{1}{343} + \frac{3}{343} + \frac{3}{343}$

[illegible]

random selections. random selections. random selections.

[illegible]

r times out of n trials. r times out of n trials. r times out of n trials.

[illegible]

Distribution of numbers with letters: [illegible]

Distribution of numbers with vowels: [illegible]

Distribution of numbers and consonants: [illegible]

Distribution of vowels with consonants: [illegible]

[illegible]

EXERCISES

[illegible]

lowed by a caramel?

[illegible]

can we very easily test this?

[illegible]

[illegible] work out these two formulæ [illegible]

[illegible] example as Ex. 4? What [illegible]

[illegible] trials or "rounds" [illegible]

[illegible] § 37. The [illegible]

[illegible] has not been introduced [illegible]

[illegible] the total number of [illegible]

[illegible]

[illegible]

$$p_x = \frac{l_{x+1}}{l_x} \qquad [IX\cdot 6]$$

[illegible]

age.

[illegible]

its value is:

$$_np_x = \frac{l_{x+n}}{l_x} \qquad [IX\cdot 8]$$

[illegible]

of this is:

$$_nq_x = \frac{l_x - l_{x+n}}{l_x} \qquad [IX\cdot 10]$$

[illegible]

bolized by $p_{x:y}$ then:

$$p_{x:y} = \frac{l_{x+1:y+1}}{l_{x:y}} = \frac{l_{x+1}}{l_x} \times \frac{l_{y+1}}{l_y} \quad [XV\ 12]$$

[illegible]

$$_{n}p_{x:y} = \frac{l_{x+n:y+n}}{l_{x:y}} \quad [XV\ 13]$$

[illegible]

EXERCISES

[illegible] Table XIX or Table XX) [illegible]

[illegible] symbols.

[illegible] silver wedding anniversary? [illegible] their fiftieth class reunion? [illegible]

MISCELLANEOUS EXERCISES

A bag containing three white [illegible] balls [illegible]

[illegible]

Ans. (a) $\frac{18}{165}$; (b) 56 to 109

[illegible] Republicans [illegible]

[illegible] Throwing six with two dice [illegible]

[illegible] Throwing at least one head [illegible]

CHAPTER X

LIFE ANNUITIES

[illegible]

(See Tables XV and XX.)

Therefore

$$= s\,\frac{D_{x+n}}{D_x}$$

[illegible] He is then alive. [illegible]

[illegible]

[illegible] with pure endowments.

EXERCISES

[illegible] amount of his endowment? [illegible]

[illegible] each of these two bequests? [illegible]

[illegible]

By probability table.

[illegible]

of a_x:

[illegible]

EXERCISES

[illegible]

annuitant.

[illegible] the age 60 is [illegible] at a slower and slower rate. [illegible] at age 70 [illegible]

[illegible]

Finally,

$$_{n|}a_x = \frac{N_{x+n}}{D_x} \quad [21.46]$$

[illegible] R dollars per [illegible]

Hence,

[illegible]

[illegible]

[illegible]

[illegible]

derivations.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Hence.

annuity due, giving:

$$\frac{N_{x+n}}{D_x}$$

It is

as follows:

$$Ra_x = R \times$$

$$R(N_x - N_{x+n})$$

used and solved

EXERCISES

[illegible]

have been [illegible] $621.63

[illegible]

year bought at age 35?

[illegible]

357

[illegible]

at four equal intervals [illegible]

[illegible]

$$[X \cdot 17] \quad a_x^{(m)} = a_x + \frac{m-1}{2m}$$

Hence,

$$[X \cdot 18] \quad R a_x^{(m)} = R\left(a_x + \frac{m-1}{2m}\right)$$

[illegible]

Hence,

and

left to the student.

EXERCISES

[illegible]

Hence,

$$_nu_x = \frac{[illegible]}{l_{x+n}}$$

multiplied by $\frac{v^{x+n}}{v^{x+n}}$

[illegible]

[illegible]

[illegible]

[illegible]

EXERCISES

[illegible]

[illegible]

[illegible] the cost, i.e., the premium [illegible]

[illegible]

from this one derivation

[illegible] at the age of purchase [illegible]

[illegible]

[illegible] table (See Table XI.) [illegible]

[illegible]

(Note that $b_{x-1} = 0$.) [illegible]

[illegible]

[illegible] meanings.

[illegible] annuity [illegible]

[illegible]

[illegible] variable rent is obtained [illegible]

[illegible]

in which

[illegible]

[illegible] is zero.

[illegible] the end of the first year [illegible]

[illegible]

$$\frac{[illegible]}{D_x}$$

[illegible] clarify it still further.

[illegible]

[illegible] ing tabulation:

[illegible]	[illegible] and [illegible] annuitant's life	[illegible]
[illegible]	Amount of payment [illegible] and [illegible] after annually	[illegible]
[illegible]	First difference [illegible] and zero there-after	[illegible]
[illegible]	Second difference [illegible] and zero there-after	[illegible]

Using formula [illegible] $X=34$ [illegible] second difference [illegible] into the formula gives:

$$\frac{[illegible]\, 12S_{[illegible]} + 4S_{[illegible]} - 2[illegible]S_{37} - [illegible]}{D_{37}}$$

[illegible]

EXERCISES

[illegible]

Ans. $4120.16

[illegible]

Ans. $525.07

[illegible]

amount of the legacy.

[illegible]

sented by $S_{x+1} \div D_x$?

[illegible]

for an annuity due.

is the first.

is the first.

is the first.

		Age + 25
		Precept
		1st difference
		2nd difference (new symbol)
		2nd difference (old symbol)

[X·24a]

[X·25a]

Note that these two formulae [illegible] student as an exercise.

[illegible]

annual rents.

[illegible]

that he will live that long?

MISCELLANEOUS EXERCISES

[illegible]

Ans. $331.76

[illegible]

Ans. $5184.53

this change?

Ans. $1672.08

[illegible]

Ans. $12,701.55

[illegible] the amount of his tax?

CHAPTER XI

LIFE INSURANCE

[illegible]

term insurance.

[illegible]

[illegible] l_x people [illegible]

[illegible] the last member is [illegible]

[illegible]

$$A_x = \frac{[illegible] + v^2 d_{x+1} + \cdots \text{ to end}}{l_x}$$

[illegible]

Hence, finally,

$$A_x = \frac{M_x}{D_x} \qquad [XI\cdot 2]$$

[illegible] insured this method gives

$$\frac{A_x}{a_x} = \frac{M_x}{D_x} \div \frac{N_x}{D_x} = \frac{M_x}{N_x} \qquad [XI \cdot 4]$$

[illegible] It is:

[illegible] annuity due.

$$G_x = FP_x = \frac{FM_x}{N_x} \qquad [XI \cdot 5]$$

[illegible] ${}_rP_x$, becomes [illegible] insurance,

$$\frac{A_x}{a_{x:\overline{r}|}} = \frac{M_x}{D_x} \div \frac{N_x - N_{x+r}}{D_x} = \frac{M_x}{N_x - N_{x+r}} \qquad [XI \cdot 6]$$

[illegible]

$${}_rG_x = \frac{FM_x}{N_x - N_{x+r}} \qquad [XI \cdot 7]$$

[illegible] Insurance [illegible]

[illegible] the term of the contract.

EXERCISES

[illegible] ended at age 60?

[illegible]

Ans. $95.52

[illegible] entail?

[illegible]

X·1. Hence,

[illegible]

insurance is:

[illegible] the [illegible] [illegible] [illegible]

$$\frac{M_x - M_{x+n}}{D_{x+n}}(1+i)_x^n \quad [XI \cdot 10]$$

This relationship [illegible] theory.

The Variable Joint [illegible] Policy [illegible]

[illegible] children [illegible] group [illegible] premium [illegible]

We must now pause [illegible] attention to a few previously defined [illegible]

[illegible]

in Table XVII.

[illegible]

[illegible] the equation becomes:

[illegible]

[illegible]

tion becomes:

[illegible]

[illegible]

becomes:

[illegible]

[illegible]

equation in the form:

[illegible]

[illegible]

tion becomes:

[illegible]

$$[illegible] \quad D_x \qquad [XI\cdot 19]$$

[illegible] the technique clearer.

[illegible] at any age thereafter [illegible] second differences:

[illegible]

[illegible] insurance.

[illegible] XI·13, giving:

$$\frac{1000(R_{40} + 2R_{41} - 7R_{42} [illegible])}{D_{40}}$$

[illegible]

EXERCISES

[illegible]

[illegible] term?

[illegible] 10 per cent [illegible] Insurance costs [illegible] $150,000 [illegible]

[illegible] level reserves,

[illegible] now do.

[illegible]

in §§ 44 and 48.

to now, age $x + t$, and summing the results. For this purpose formula XI·8 should be used with $n = 1$ in each case giving:

$$A^1_{x\overline{1|}} = \frac{C_x}{D_x},\ A^1_{x+1\overline{1|}} = \frac{C_{x+1}}{D_{x+1}},\ A^1_{x+2\overline{1|}} = \frac{C_{x+2}}{D_{x+2}} \cdots \text{ up to } A^1_{x+t\overline{1|}} = \frac{C_{x+t}}{D_{x+t}}$$

Each A^1 is dated at the beginning of the year of its coverage. To bring these dated values to now, age $x + t$, each must be multiplied by the proper survivorship-interest factor, the first by $(1 + i)^t_x$, the second by $(1 + i)^{t-1}_{x+1}$ and so on to the last, which is multiplied by 1 only. This gives, upon summing these results:

$$\frac{C_x + C_{x+1} + C_{x+2} + \cdots + C_{x+t}}{D_{x+t}} = \frac{M_x - M_{x+t+1}}{D_{x+t}} = {}_tk_x$$

a quantity which has already been defined in § 47 and worked out in equation XI·18.

A moment's thought and reference to § 45 will convince one that the present value at age $x + t$ of the protection benefits of the full-life insurance from that age onward is A_{x+t}. Hence the total present value at age $x + t$ of both past and future benefits of a full-life insurance of \$1 is seen to be ${}_tk_x + A_{x+t}$.

Turning now to the cost to the insured of these benefits it is evident that, at age $t + x$, he has already paid a temporary life annuity due of annual rent equal to the net annual premium of the \$1 full-life insurance, or $P^1_{x\overline{t|}}$ and the present value at age $x + t$ is $P^1_x\, \mathrm{a}_{x\overline{t|}}\, (1 + i)^t_{x\overline{t|}}$ which may be written more briefly as $P_x\, {}_tu_x$ as is seen in equation X·22. Moreover he still has to pay a full-life annuity due as of this date and of rent P_x, the present value of which would therefore be $P_x \mathrm{a}_{x+t}$. Hence the total cost of or present value of the cost of the full-life insurance of \$1 is the sum of these two like-dated annuities:

$$P_x\, {}_tu_x + P_x\, \mathrm{a}_{x+t}$$

But equation XI·21 tells us that the value of the benefits equals the cost of any insurance policy, including a \$1 policy. Hence we get:

$$P_x\, {}_tu_x + P_x\, \mathrm{a}_{x+t} = {}_tk_x + A_{x+t}{}^* \qquad \text{[XI·22]}$$

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[*] The student should note carefully that while ${}_tu_x$ and ${}_tk_x$ vary with t and x only, P, A and a depend upon the type of insurance and annuity involved.

[illegible]

equations:

$${}_tV_x = P_x\,{}_tu_x \quad [\text{XI. } 23]$$

and

$${}_tV_x = A_{x+t} - [\text{XI. } 24],$$

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The *initial reserve* of a given policy is the reserve at the beginning of a policy year just after the premium has been paid. Hence the initial reserve of the tth policy year is the *terminal reserve* of the $t-1$ year plus the premium (net) paid at the beginning of the tth year.

The *mean reserve* is the average between the initial and the terminal reserves of any given policy year. State departments require insurance companies to file at the end of each calendar year a report showing among other things their total reserves on all policies outstanding. In preparing this statement all policies issued during the calendar year are assumed to have been issued on July 1, which would make the policy year begin on July 1. The reserves on January 1 would therefore be the mean reserves of the policy year.

The *net amount of risk* for a given policy in the tth year is the difference between the face of the policy and the terminal reserve for that year. Per $1 of policy it would be $1 - {}_tV_x$.

The surrender values when one wishes for any reason to discontinue his insurance are the values of the options by which he may discontinue his insurance. There are usually three options among which one may choose when he surrenders his policy:

1. *Cash surrender value,* an amount payable immediately upon surrender of the Policy; or

2. *Paid-up insurance,* which consists of a fully paid-up policy for a smaller amount but otherwise like the original policy, or

3. *Extended insurance,* a fully paid-up policy for the original amount but for a temporary period.

Usually when the insured makes no option the third goes into effect automatically.

The cash surrender value is found by making a deduction called the *surrender charge* from the terminal reserve. State laws provide the maximum surrender charge and it is usually $25 per $1000. The charge is usually made to decrease as the age of the policy increases until it finally discontinues entirely after a period which varies from 5 to 20 years.

The paid-up insurance is for a face value which the cash surrender value would purchase upon the original terms of the policy.

The term of the extended term insurance is so determined that the present value of the term insurance equals the cash surrender value of the original policy.

EXERCISES

1. By turning the two equations XI·23 and XI·24 into commutation symbols and performing the appropriate algebraic transformations, reduce them to identical forms and in this way prove that they are really two ways of finding the same value, the terminal reserve at the end of any year.

2. Show that the only change needed in the retrospective formula, XI·23, to make it applicable to term insurance or endowment insurance is in the factor representing the net annual payment. Write the form needed to compute the terminal reserve for term insurance and that for endowment insurance, symbolizing the first by ${}_tV^1_{x\overline{n}|}$ and the second by ${}_tV_{x\overline{n}|}$. When the premium payments are limited to a shorter term, r, than the coverage, n, this fact is symbolized by writing these symbols ${}_{t,r}V^1_{x\overline{n}|}$ and ${}_{t,r}V_{x\overline{n}|}$ respectively.

3. Make the changes that you consider necessary in the prospective equation, XI·24, to make it applicable to these same two types of insurance and then prove your results by first turning them over into commutation symbols and then into the following forms: ${}_tV^1_{x\overline{n}|} = A^1_{x+t\,\overline{n-t}|} - P^1_{x\overline{n}|}\,a_{x+\,\overline{n-t}|}$ and ${}_tV_{x\overline{n}|} = A_{x+t.\overline{n-t}|} - P_{x\overline{n}|}\,a_{x+t\overline{n-t}|}$

4. Find the terminal reserve or the net level reserve of the third year by both the retrospective and the prospective methods of the policy treated in Ex. 1 on page 188. Be sure that you use the proper form of the terminal-reserve equation to be applicable to the type of insurance of the problem under consideration.

MISCELLANEOUS EXERCISES

1. Find by both the retrospective and the prospective methods the terminal reserve of the fifth policy year of a full-life insurance of $1 coverage taken out at age 30 (see page 193). *Ans.* .050575

2. In order to study the way the terminal reserve builds up in the case of a full-life insurance find the terminal reserve at 15-year intervals on a $10,000 policy taken out at age 30. Use the two methods in alternation and carry the computations through to the end of the table and put your results into tabular form for reference. *Ans.* (*a*) $1794.74; (*c*) $6990.50

3. If the policy in Ex. 2 had been a 20-premium 20-year term insurance instead of a full life, what difference would it have made in the terminal reserves? Tabulate them for 5-year intervals.

Ans. (*b*) $104.15; (*c*) $101.17

4. If the policy in Ex. 3 were again changed, this time to a 20-year endowment policy, find and tabulate the terminal reserves for 5-year intervals.

Ans. (*a*) $1778.08

5. In addition to the retrospective method given by equation XI·23 there is another which builds the reserve of any policy year from the one immediately before it. Expressed verbally, the method consists of accumulating the terminal reserve of the previous year for one year, adding the total number of net annual premiums and subtracting the value of the death claims of the current year. Put this verbal formula for ${}_{t+1}V_x$ into literal form and see what you can do towards simplifying the result. This formula, in its simplest form, is known

as *Fackler's Accumulation Formula.* His form of it is ${}_{t+1}V_x = {}_1u_{x+t}({}_tV_x + P_x) - k_{x+t}$.

6. Prove that $M_x = vN_x - N_{x+1}$.

7. Prove that ${}_tV_x = (P_{x+t} - P_x)\mathrm{a}_{x+t}$.

8. Prove that $A_x = P_x$ divided by $P_x - d$.

9. Prove that $A_x = v\mathrm{a}_x - a_x$.

10. Prove that $A_x = 1 - d\mathrm{a}_x$.

11. If the insured in Ex. 3 had wished to pay up his policy by paying 10 annual premiums instead of 20, what would be the value of the net annual premium? *Ans.* \$154.51

12. Work out the terminal reserves for 5-year intervals and tabulate them in the limited-payment term insurance of Ex. 11. *Ans.* \$878.72

13. Find the terminal reserve of the twentieth year on a 20-payment life policy for \$1000 issued at age 25. Note and account for the fact that it equals the net single premium at age 45. *Ans.* \$456

14. Find the terminal reserve of the twentieth policy year on 20-year term insurance for \$1000 issued at age 25. *Ans.* \$0.0448

15. Find the fifth-year terminal reserve on a 10-year term policy for \$1000 issued at age 22. *Ans.* 82 cents

16. What is the net single premium at age 35 for a whole-life insurance in which the beneficiary receives \$1000 at the death of the policyholder and \$1000 per year for the next 9 years. *Ans.* \$3189.69

17. Recently insurance has been combined with an annuity contract as well as a pure endowment. Find the net single premium and the net annual premium that must be paid for 25 years by a man aged 40 to provide for the following benefits: (a) a \$5000 pure endowment to be paid at age 65; (b) a \$5000 20-payment full-life insurance and (c) a life annuity of \$1000 annually to begin payment on the insured's sixty-fifth birthday.

Ans. Net annual = \$329.15

TABLES

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TABLE VII. THE PRESENT VALUE OF 1 DUE $\frac{d}{360}$ OF A YEAR HENCE

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$v^{\frac{d}{360}} = (1+i)^{-\frac{d}{360}}$

TABLE VII. THE PRESENT VALUE OF 1 DUE $\frac{d}{360}$ OF A YEAR HENCE

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TABLE VII. THE PRESENT VALUE OF 1 $\frac{d}{360}$ OF A YEAR HENCE

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TABLE VII. THE PRESENT WORTH OF $1 $\frac{d}{360}$ OF A YEAR HENCE

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TABLE VII. THE PRESENT VALUE OF $\frac{d}{360}$ OF A YEAR HENCE

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CERTAIN EFFECTIVE RATES

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.007	.0149	.0147 783	0.0150
.008	.0173	.0171 990	0.0175
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.011	.0222	.0220 049	0.0225
.012	.0246	.0243 902	0.0250
.014	.0295	.0291 262	0.0300
.017	.0344	.0338 164	0.0350
.019	.0392	.0384 615	0.0400
.022	.0410	.0430 622	0.0450
.024	.0487	.0476 190	0.0500
.029	.0582	.0566 038	0.0600
.034	.0676	.0654 206	0.0700
.039	.0769	.0740 741	0.0800

[illegible]

	3		1	2	3		1	2	3		1	2
Six Times this (1+p) Product		¼	Product of Columns 1 and 2	6 X Product	Six Times this (1+p) Product	¼	Product of Columns 1 and 2	6 X Product	Six Times this (1+p) Product	¼	Product of Columns 1 and 2	6 X Product
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

utation Symbols   Commutation Symbols   Commutation Symbols   Commu

rience 3½% American Experience 3½% American Experience 3½% American Expe

[illegible]

$\phi_x = \frac{10,000}{D_x}$	Age x	$\sigma_x = 10,000 \frac{S_x}{D_x}$	Age x	$\phi_x = \frac{10,000}{D_x}$	Age x	$\sigma_x = 10,000 \frac{S_x}{D_x}$	Age x	$\phi_x = \frac{10,000}{D_x}$	Age x	$\sigma_x = 10,000 \frac{S_x}{D_x}$	Age
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible] $+\frac{M_z}{D_z}$, [illegible]

A_g^p		[illegible]	k_z	C_{kz}
[illegible]	[illegible]	[illegible]	[illegible]	5
[illegible]	[illegible]	[illegible]	[illegible]	4
51.77[illegible]	.97[illegible]	5.08	0.2547	4
[illegible]	[illegible]	[illegible]	[illegible]	6
[illegible]	[illegible]	[illegible]	[illegible]	1
0.8902	2	[illegible]	[illegible]	167
[illegible]	[illegible]	[illegible]	[illegible]	8
[illegible]	[illegible]	[illegible]	[illegible]	7
43.[illegible]	[illegible]	92.47	0.2726	1
[illegible]	[illegible]	[illegible]	[illegible]	9
[illegible]	[illegible]	[illegible]	[illegible]	1
0.6515	3	[illegible]	[illegible]	117
[illegible]	[illegible]	[illegible]	[illegible]	7
[illegible]	[illegible]	[illegible]	[illegible]	0
37.[illegible]	[illegible]	[illegible]	0.2939	9
[illegible]	[illegible]	[illegible]	[illegible]	3
[illegible]	[illegible]	[illegible]	[illegible]	5
0.7122	3	[illegible]	[illegible]	022
[illegible]	[illegible]	[illegible]	[illegible]	3
[illegible]	[illegible]	[illegible]	[illegible]	1
31.1387	[illegible]	74.03	0.3194	7
[illegible]	[illegible]	[illegible]	[illegible]	2
[illegible]	[illegible]	[illegible]	[illegible]	7
0.7690	4	[illegible]	[illegible]	851
[illegible]	[illegible]	[illegible]	[illegible]	2
[illegible]	[illegible]	[illegible]	[illegible]	8
25.87[illegible]	[illegible]	[illegible]	0.3497	6
[illegible]	[illegible]	[illegible]	[illegible]	6
[illegible]	[illegible]	[illegible]	[illegible]	9
0.8206	4	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	5
[illegible]	[illegible]	[illegible]	[illegible]	5
21.[illegible]	[illegible]	[illegible]	0.3856	0
[illegible]	[illegible]	[illegible]	[illegible]	9
[illegible]	[illegible]	[illegible]	[illegible]	3
0.[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	3
[illegible]	[illegible]	[illegible]	[illegible]	9
17.[illegible]	[illegible]	[illegible]	0.4276	2
[illegible]	[illegible]	[illegible]	[illegible]	1
[illegible]	[illegible]	[illegible]	[illegible]	8
0.[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	8
[illegible]	[illegible]	[illegible]	[illegible]	8
[illegible]	[illegible]	[illegible]	0.4787	2
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

1000P/s_z	Age z	1000P/s_z
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
11,1889270	4 11	45.3753 21
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
82.3223 20	5748	[illegible] 83
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
12,1078017 1102	52	46.4894 60
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
97.0470 31	[illegible]	[illegible] 82
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	53	47.8976 46
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
117.3125 2	[illegible]	[illegible] 319
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	55	49.8909 53
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
145.3963 5	[illegible]	[illegible] 676
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	00	52.0058 67
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
155.5403 3	[illegible]	[illegible] 049
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	25	55.0393 32
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
257.1490 2	[illegible]	[illegible] 042
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	43	59.0801 99
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
390.1121 0	[illegible]	[illegible] 051
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	02	64.2489 20
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]

[illegible]

[illegible]

$\frac{D_x}{D_{x+n}}$ $(1+i)^n_x = \frac{D_x}{D_{x+n}}$ $(1+i)^n_x = \frac{D_x}{D_{x+n}}$ $(1+i)^n_x =$

[illegible]

STILL LIVING—(*Continued*)

$$(1+i)^n_x = \frac{D_x}{D_{x+n}}$$

$$\$10{,}000\,{}_nE_x = \$10{,}000 \times \frac{D_{x+n}}{D_x}$$

(Continued)

$$(1+i)^n_x = \frac{D_x}{D_{x+n}}$$

(Continued)

$$(1+i)^n_x = \frac{D_x}{D_{x+n}}$$

LIVING—(*Continued*)

$$(1 + i)^n_x = \frac{D_x}{D_{x+n}}$$

(*Continued*)

$$\$10{,}000 \times \frac{D}{D_x}$$

(Continued)

$$(1+i)^n_x = \frac{D_x}{D_{x+n}}$$

STILL LIVING—(*Continued*)

$$(1+i)^n_x = \frac{D_x}{D_{x+n}}$$

[illegible]

STILL LIVING—(*Continued*)

$$\$10{,}000\,v^n_x = \$10{,}000 \times \frac{D_{x+n}}{D_x}$$

[illegible]

(Continued)

$$(1 + i)_x^n = \frac{D_x}{D_{x+n}}$$

[illegible]

LIVING—(*Continued*)

$$(1+i)^{n}_{x} = \frac{D_x}{D_{x+n}}$$

[illegible]

(*Continued*)

$$\$10{,}000\,v^{n}_{x} = \$10{,}000 \times \frac{D_{x+n}}{D_x}$$

[illegible]

(Continued)

$$(1+i)_x^n = \frac{D_x}{D_{x+n}}$$

[illegible]

STILL LIVING (*Continued*)

$$[(1+i)^n_x] = \frac{D_x}{D_{x+n}}$$

[illegible]

[illegible]

LIVING—(*Continued*)

$$\$10{,}000 \times \frac{D_{x+n}}{D_x}$$

[illegible]

(*Continued*)

$$(1+i)^{n}_{x} = \frac{D_x}{D_{x+n}}$$

(Continued)

$$(1+i)^n_x = \frac{D_x}{D_{x+n}}$$

(Continued)

$$(1+i)^n_x = \frac{D_x}{D_{x+n}}$$

(Continued)

$$(1+i)^n_x = \frac{D_x}{D_{x+n}}$$

[illegible]

$$\$10{,}000 \times \frac{D_{x+n}}{D_x} = \$10{,}000\, v^n_x$$

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3	52			0.1047 381	
3	51	0.0959 159	0	0.6949 108	
[illegible]	[illegible]	0.6621 629	8	2.[illegible]	
[illegible]	[illegible]	2.[illegible]		476.[illegible]	
[illegible]	[illegible]	476.[illegible]		[illegible]	

[illegible]

(Continued)

$$(1 + i)_x^n = \frac{D_x}{D_{x+n}}$$

[illegible]

$\times \frac{D_{x+n}}{D_x}$ \$10,000 v_x^n = \$10,000 $\times \frac{D_{x+n}}{D_x}$ \$10,000 v_x^n = \$10,000 $\times \frac{D_{x+n}}{D_x}$ \$10,000 v_x^n = \$10,000

0.1382 211 0

0.1396 9769 8 0.9469 049 2

0.8946 9216 5 3.4013 [illegible]

3.8104 [illegible]

[illegible]

(Continued)

$$(1+i)_x^n = \frac{D_x}{D_{x+n}}$$

(Continued)

$$(1+i)_x^n = \frac{D_x}{D_{x+n}}$$

(Continued)

$$(1+i)_x^n = \frac{D_x}{D_{x+n}}$$

[illegible]

$$\times \frac{D_x}{D_x} \$10,000 v_x^n = \$10,000 \times \frac{D_x}{D_x} \$10,000 v_x^n = \$10,000 \times \frac{D_x}{D_x} \$10,000 v_x^n = \$10,0$$

0. 2015 793 6 | 64
0. 1881 517 4 | 1. 3631 5[illegible] | 63
1. 2758 435 | 4. 9678 [illegible]

[illegible]

STILL LIVING—(*Continued*)

$$(1+i)^n = \frac{D_x}{D_{x+n}}$$

(*Continued*)

$$\$10{,}000 \times \frac{D_{x+n}}{D_x}$$

[illegible]

[illegible]

$$\ldots 00 \times \frac{D_{x+n}}{D_x} \quad \$10{,}000\,v_x^n = \$10{,}000 \times \frac{D_{x+n}}{D_x} \quad \$10{,}000\,v_x^n = \$10{,}000 \times \frac{D_{x+n}}{D_x} \quad \$10{,}000\,v_x^n = \$10{,}\ldots$$

[illegible]

(Continued) Still Living—(Continued)

$$(1 + i)_{\overline{n}|} = \frac{D_x}{D_{x+n}}$$

[illegible]

(Continued)

$$\$10{,}000 \times \frac{D_{x+n}}{D_x}$$

[illegible]

[illegible]

$000 \times \frac{D_{x+n}}{D_x}$ \$10,000v^n_x = \$10,000 $\times \frac{D_{x+n}}{D_x}$ \$10,000v^n_x = \$10,000 $\times \frac{D_{x+n}}{D_x}$ \$10,000v^n_x = \$10,

[illegible]

STILL LIVING—(*Continued*)

$$(1+i)^n_x = \frac{D_x}{D_{x+n}}$$

[illegible]

(*Continued*)

$$\$10{,}000 \times \frac{D_{x+n}}{D_x}$$

[illegible]

[illegible]

$$\times \frac{D_{x+t}}{D_x} \quad \$10{,}000 v_x^n = \$10{,}000 \times \frac{D_{x+t}}{D_x} \quad \$10{,}000 v_x^n = \$10{,}000 \times \frac{D_{x+t}}{D_x} \quad \$10{,}000 v_x^n = \$10{,}00$$

16	17	14	16	18	16	17	14	16	18	16	17	14	16	18
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible]

(Continued) STILL LIVING—(Continued) STILL LIVING—(Continued) STILL LIVING—

$$\frac{D_x}{D_{x+n}} \qquad (1+i)_x^n = \frac{D_x}{D_{x+n}} \qquad (1+i)_x^n = \frac{D_x}{D_{x+n}} \qquad (1+i)_x^n =$$

[illegible]

[illegible]

(Continued) (Continued) (Continued)

$$000 \times \frac{D_{x+n}}{D_x} \quad \$10{,}000 v_x^n = \$10{,}000 \times \frac{D_{x+n}}{D_x} \quad \$10{,}000 v_x^n = \$10{,}000 \times \frac{D_{x+n}}{D_x} \quad \$10{,}000 v_x^n = \$10,$$

[illegible]

$$\times \frac{D_{x+n}}{D_x} \$10{,}000 v_x^n = \$10{,}000 \times \frac{D_{x+n}}{D_x} \$10{,}000 v_x^n = \$10{,}000 \times \frac{D_{x+n}}{D_x} \$10{,}000 v_x^n = \$10{,}000$$

[illegible]

STILL LIVING—(*Continued*)

$$(1 + i)^n_x = \frac{D_x}{D_{x+n}}$$

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	.777	7	2423
15	6	9442.770	50	3112.963
16	48	2388.40		

$[(1 + i)^{[illegible]}_{[illegible]} = 62\ 0595.81]$

[illegible]

(*Continued*)

$$\$10{,}000\,{}_nE_x = \$10{,}000 \times \frac{D_{x+n}}{D_x}$$

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		1380.2619	8	94
254.4979	5	2568.3381	1	[illegible]
907.5798	6	3533.[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible]

[illegible] ($10,000 [illegible]

$$000 \times \frac{D_{x+n}}{D_x} [illegible] \$10{,}000 v^n_x = \$10{,}000 \times \frac{D_{x+n}}{D_x} [illegible] \$10{,}000 v^n_x = \$10{,}000 \times \frac{D_{x+n}}{D_x} [illegible] \$10{,}000 v^n_x = \$10{,}[illegible]$$

[illegible]

[illegible]

FALLS

[illegible]

simple directions:

[illegible]

Remember that leap years are [illegible]

[illegible]

Transfer March 14, 1920, [illegible]

[illegible]

Saturday.

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